Octet Enhancement in the B and Δ Supermultiplets*

ROGER F. DASHEN AND STEVEN C. FRAUTSCHI†
California Institute of Technology, Pasadena, California
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The bootstrap theory of octet enhancement is described in general, and applied in particular to strong and electromagnetic mass splittings within the $J^P=\frac{1}{2}^+$ baryon octet and $J=\frac{3}{2}^+$ decuplet. The results are in good agreement with experiment.

I. INTRODUCTION

It appears that electromagnetic mass splittings within the baryon isospin multiplets transform mostly like the third component of an octet.[1,2] It also appears that nonleptonic weak decays are dominantly octet in view of the $|\Delta\mathbf{I}|=\frac{1}{2}$ rule.[3] These phenomena are second order in the electromagnetic and weak currents, respectively. If as is generally assumed, both the electromagnetic and weak currents have octet transformation properties, then phenomena of second order in these currents will generally contain both **8** and **27** terms, and the observed octet dominance requires some explanation.

Another unexplained fact is the success of the Gell-Mann–Okubo sum rule[4,5] which seems to represent the main features of strong mass splittings among members of a supermultiplet, indicating that the splittings transform like the eighth component of an octet. Several views on strong symmetry breaking have been advanced. Ne'eman,[6] for example, introduces a "fifth interaction" in which the coupling of a singlet vector meson to the hypercharge and baryon currents causes the symmetry breaking, in close analogy to the coupling of photon and electric current. The terms of second order in the currents, which generate the strong mass splitting in Ne'eman's theory, contain both **8** and **27** and the observed octet dominance requires an explanation just as it did for second-order electromagnetic and weak phenomena. An alternative suggestion is that strong symmetry breaking occurs spontaneously[7] in the bootstrap theory. This proposal puts strong $SU(3)$ symmetry violations on a different footing than electromagnetic and weak violations, but an explanation of why the symmetry breaking transforms like a member of an octet is still needed.

Extrapolating from the empirical evidence just cited, Coleman and Glashow[1] have suggested that octet dominance is a very general feature of $SU(3)$ symmetry breaking. This led them to the further suggestion that there should be a common dynamical cause underlying octet dominance in strong, electromagnetic, and weak symmetry violations.

In a recent letter[8] we discussed this suggestion that octet dominance is a general phenomenon with a common dynamical origin. In place of the "tadpole" mechanism which Coleman and Glashow specifically advocated as the cause, however, we employed ideas of Cutkosky and Tarjanne[7,9] and proposed a bootstrap mechanism for octet enhancement. It was explained how the mechanism can apply universally to strong, electromagnetic, and weak symmetry violations no matter which view of strong $SU(3)$ symmetry breaking is adopted—spontaneous breakdown or a "fifth interaction."

A review of the bootstrap mechanism for octet enhancement is provided in Sec. II of the present paper. The method seems capable of explaining not only the general existence of octet dominance but also the ratios of the symmetry-breaking octet terms in the baryon octet, $J=\frac{3}{2}^+$ decuplet, and so forth. Even without detailed dynamical calculations, if the existence of octet enhancement is accepted, its universal nature leads to predictions relating the pattern of strong, electromagnetic, and weak symmetry violations, and we include in Sec. II experimental evidence which supports these predictions as well as some discussion of how the predictions can be used phenomenologically.

The rest of the present paper is concerned with detailed dynamical calculations of the bootstrap mechanism. $SU(3)$-symmetric solutions of the strong interaction bootstrap are taken as input. Symmetry-breaking perturbations are studied with the S-matrix perturbation theory which the authors developed in a previous paper[10] and subsequently adapted[11] to problems involving symmetry groups and coupled channels.

In Sec. III the convergence of the dispersion integrals appearing in the S-matrix perturbation theory is discussed. It is made plausible, with the help of some considerations in Appendix A, that the dispersion

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[3] N. Cabibbo, Phys. Rev. Letters **12**, 62 (1964).
[4] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962).
[5] S. Okubo, Progr. Theoret. Phys. (Kyoto) **27**, 949 (1962).
[6] Y. Ne'eman, Phys. Rev. **134**, B1355 (1964).
[7] R. E. Cutkosky and P. Tarjanne, Phys. Rev. **132**, 1355 (1963). The general nature of the bootstrap mechanism was stressed by Cutkosky at the Pasadena meeting of the American Physical Society, December 1963 (unpublished speech).
[8] R. Dashen and S. Frautschi, Phys. Rev. Letters **13**, 497 (1964).
[9] Extensive work along related lines has been done by R. H. Capps; e.g., Phys. Rev. **134**, B1396 (1964).
[10] R. Dashen and S. Frautschi, Phys. Rev. **135**, B1190 (1964).
[11] R. Dashen and S. Frautschi, Phys. Rev. **137**, B1318 (1965) (preceding paper).

integrals converge more rapidly if the strongly interacting particles are bound states or resonances, whereas when elementary particles are present the integrals converge more slowly if at all. Thus the prospect for practical success of our calculations seems closely tied to the hypothesis that the strongly interacting particles are composite. Assuming that they are, we feel justified in keeping only low-mass states and nearby singularities in our bootstrap calculations.

In Sec. IV we turn specifically to the bootstrap theory of the $J=\frac{1}{2}^+$ B octet and $J=\frac{3}{2}^+$ Δ decuplet. We choose this case for consideration rather than, say, a pseudoscalar-meson–vector-meson reciprocal bootstrap because the physical basis of the latter is dubious —in particular, it is not well known that channels the pseudoscalar meson are most strongly coupled to, and perturbations on such a poorly understood situation are likely to be unreliable. On the other hand, the B-Δ reciprocal bootstrap is probably reliable, and Sec. IV together with Appendix B is mainly concerned with the consequences of keeping only low-mass states and nearby singularities in this case. Static kinematics then become approximately correct and one finds that mass shifts in the $J=0^-$ Π octet, and coupling shifts, may have relatively little effect on the B and Δ mass shifts.

This simplification, and some further simplifications discussed in Appendix C, are utilized in the calculation of $SU(3)$ symmetry violations for the B and Δ masses, carried out in Sec. V. The calculation yields octet dominance, and the numerical results compare very favorably with the pattern of experimental strong and electromagnetic mass splittings, as already mentioned in our letter.[8]

In Sec. VI we discuss higher order symmetry breaking, such as the effects of strong symmetry violations on a calculation of the B and Δ electromagnetic mass shifts. This discussion clarifies the relation between the $SU(3)$ B-Δ reciprocal bootstrap of Sec. V and the usual $SU(2)$ $N-N^*$ reciprocal bootstrap (treated in Sec. VII and Appendix D). As a byproduct, we find that whereas $SU(3)$ supermultiplets are possibly unstable against spontaneous symmetry breakdown, the $SU(2)$ multiplets N and N^* are stable, as might be expected from the work of Abers, Zachariasen, and Zemach.[12]

II. BOOTSTRAP MECHANISM FOR OCTET ENHANCEMENT

We begin by assuming the bootstrap theory of strong interactions, describing each strongly interacting particle as a bound or resonant state of strongly interacting particles. Unstable particles and stable ones are treated on the same footing. If $SU(3)$ symmetry held exactly, each supermultiplet would appear as a set of degenerate poles in the various scattering amplitudes with appropriate quantum numbers.

[12] E. Abers, F. Zachariasen, and C. Zemach, Phys. Rev. **132**, 1831 (1963).

Now for purposes of describing the octet enhancement mechanism, we consider the specific case of electromagnetic corrections of order e^2. For simplicity, we ignore the effects of strong $SU(3)$ violations on the electromagnetic corrections (they will be considered in Sec. VI). The electromagnetic interaction causes shifts in the positions of the poles in each supermultiplet from their original common value, and shifts in the residues of the poles from their original values as given by $SU(3)$ symmetry. The formalism for treating the shifts has been developed in the accompanying paper.[11] The shifts in position of the poles representing a supermultiplet provide a mass shift matrix δM for that supermultiplet. The shifts in residues of the poles provide shifts $\delta\Gamma$ in the coupling of the supermultiplet represented by the poles to the external particles in the amplitude.

The mass and coupling shifts of bound supermultiplets follow in part from direct electromagnetic effects such as photon exchange. We call these effects the driving terms D; in principle, they include all diagrams in which an intermediate state in any channel contains a particle which was not present in the bootstrapped strong interaction (i.e., a photon in the electromagnetic case). In the bootstrap theory, mass and coupling shifts of the bound states also appear in response to electromagnetic mass and coupling shifts of the exchanged and external particles which were already present in the bootstrapped strong interaction. Combining these effects, we obtain, to order e^2 in the electromagnetic perturbation, relations of the form

$$\frac{\delta M_{ij}{}^{\alpha}}{M^{\alpha}}=\sum_{\alpha',k,l}A_{ij,kl}{}^{\alpha\alpha'}\frac{\delta M_{kl}{}^{\alpha'}}{M^{\alpha'}}+\sum_{\beta',k,l,m,\cdots}A_{ij,klm\cdots}{}^{\alpha\beta'}\delta\Gamma_{klm\cdots}{}^{\beta'}+D_{ij}{}^{\alpha}, \quad (2.1)$$

$$\delta\Gamma_{ijk\cdots}{}^{\beta}=\sum_{\alpha',m,n}A_{ijk\cdots,mn}{}^{\beta\alpha'}\frac{\delta M_{mn}{}^{\alpha'}}{M^{\alpha'}}+\sum_{\beta',m,n,o,\cdots}A_{ijk\cdots,mno\cdots}{}^{\beta\beta'}\delta\Gamma_{mno\cdots}{}^{\beta'}+D_{ijk\cdots}{}^{\beta}. \quad (2.2)$$

Here, we define the Γ's to be dimensionless, so that A is dimensionless. On $\delta M_{ij}{}^{\alpha}$ the label α runs over the different supermultiplets such as B, Δ, and Π while i and j run over the members of each supermultiplet. On $\delta\Gamma_{ijk\cdots}{}^{\beta}$ the label β runs over the different vertices such as $B\Delta\Pi$ and $BB\Pi$ while i runs over the members of the first supermultiplet participating in each vertex, j over the members of the second supermultiplet, and so forth.

As described in the accompanying paper,[11] it is most convenient to expand each δM^{α} and $\delta\Gamma^{\beta}$ in irreducible representations of $SU(3)$. The physical reason is that in our order e^2 equations, the A coefficients contain no $SU(3)$ violation so that they connect **8** violations δM_8 or $\delta\Gamma_8$ only to **8** violations, **27** violations only to **27** violations, and so forth. Thus, one obtains a simplified set of equations

$$\delta M_{\lambda,n}{}^{\alpha}/M^{\alpha}=\sum_{\alpha'} A_{\lambda}{}^{\alpha\alpha'}(\delta M_{\lambda,n}{}^{\alpha'}/M^{\alpha'}) +\sum_{\beta'} A_{\lambda}{}^{\alpha\beta'}\delta\Gamma_{\lambda,n}{}^{\beta'}+D_{\lambda,n}{}^{\alpha}, \quad (2.3)$$

$$\delta\Gamma_{\lambda,n}{}^{\beta}=\sum_{\alpha'} A_{\lambda}{}^{\beta\alpha'}(\delta M_{\lambda,n}{}^{\alpha'}/M^{\alpha'}) +\sum_{\beta'} A_{\lambda}{}^{\beta\beta'}\delta\Gamma_{\lambda,n}{}^{\beta'}+D_{\lambda,n}{}^{\beta}, \quad (2.4)$$

where λ is the supermultiplicity of the violation and $n=1,\ \cdots\lambda$ is the component of that supermultiplet utilized by the violation. It is crucial that the A coefficients, containing no $SU(3)$ violation, connect n only to the same n and are independent of n. The indices α and β now run not only over the various supermultiplets and vertices, but also over the independent irreducible representations with supermultiplicity λ that occur within each mass matrix or vertex. For example, the $\mathbf{8}\times\mathbf{8}$ B mass matrix contains two independent octet terms, and the $\mathbf{8}\times\mathbf{8}\times\mathbf{8}$ $BB\Pi$ vertex contains a number of independent octet terms.

We can rewrite Eqs. (2.3) and (2.4) in a matrix form

$$\begin{bmatrix} \\ 1-A_{\lambda} \\ \\ \end{bmatrix} \begin{pmatrix} \delta M_{\lambda,n}{}^{1}/M^{1} \\ \delta M_{\lambda,n}{}^{2}/M^{2} \\ \vdots \\ \delta\Gamma_{\lambda,n}{}^{1} \\ \vdots \end{pmatrix} = \begin{pmatrix} D_{\lambda,n}{}^{1} \\ D_{\lambda,n}{}^{2} \\ \vdots \\ \cdot \\ \vdots \end{pmatrix}. \quad (2.5)$$

Now in terms of Eq. (2.5), the problem set forth at the beginning of this paper was that D_8 and D_{27} are of comparable magnitude but δM_8 emerges as the dominant electromagnetic mass splitting. It is easy to see how this can occur in our formalism. First, consider what would happen if there were no $\delta\Gamma$'s and only one δM. We could then solve for δM

$$\delta M_{\lambda,n}/M=(1/(1-A_{\lambda}))D_{\lambda,n} \quad (2.6)$$

and the octet shift would be preferentially enhanced if A_8 were near unity and A_{27} were far from unity. In the full matrix problem of Eq. (2.5), one has to consider the eigenvalues of A_{λ}. If A_8 has one (or more) eigenvalues near unity, then $\delta M_{8,n}$ and $\delta\Gamma_{8,n}$ will contain a large term multiplying the associated eigenvector(s). If the matrix A_{27} lacks eigenvalues near unity, the octet is preferentially enhanced.

Now suppose that we accept for a minute Ne'eman's mechanism[6] for strong symmetry breaking, so that strong, electromagnetic, and weak symmetry breaking all have definite driving terms and are all on the same basis.[13] In this case, the mechanism just described for electromagnetic octet enhancement applies generally to all violations of $SU(3)$ which are linear in the masses and couplings, since the eigenvalues and eigenvectors of A_8 and A_{27} are independent of the driving term D and independent of the axis n along which the violation lies in $SU(3)$ space. Thus, barring the unlikely case that the driving term has no component along the enhanced eigenvector, isospin-conserving strong mass and coupling shifts ($n=8$), isospin-violating electromagnetic shifts ($n=3$), and (P- and C-conserving) weak shifts should all exhibit octet enhancement and all lie along the same eigenvector (i.e., all have the same ratios among the independent octet matrices such as the Δ mass term, the D and F baryon mass terms, and the various $BB\Pi$ coupling terms).[14]

If spontaneous $SU(3)$ violation rather than a "fifth interaction" is responsible for strong symmetry breaking, then strong symmetry breaking requires a separate discussion based on the form of Eq. (2.5) before it is inverted. In this case, $D_{\rm strong}=0$ and the enhanced eigenvalue must equal unity in the linear approximation. When higher orders are included, however, the eigenvalue is no longer required to be exactly one. Thus the same condition that makes octet enhancement occur (an eigenvalue of A_8 near unity) makes spontaneous breakdown conceivable. Our present methods are not strong enough to decide which mechanism is actually taking place, but in either case the strong mass and coupling shifts are expected to lie along the eigenvector associated with eigenvalue $A_8\approx 1$.

Parity- and charge-conjugation-violating terms in the weak interactions can be studied with the same techniques. There is no mass shift associated with these terms, since they connect only states which were not connected by the P- and C-conserving strong interactions. The inability of "off-diagonal" terms to provide a lowest order mass shift, which we are using here, is familiar from the perturbation expansion of Schrödinger theory and has been discussed in an S-matrix context in the accompanying paper.[11] Thus, for example, the A_8 matrix for P and C violations connects only the coupling shifts $\delta\Gamma_{8,n}{}^{\beta}$, where β runs over the independent octets of P- and C-violating vertices. It is obvious that this A matrix will have different eigenvalues and eigenvectors than the A matrix which connects P- and C-conserving shifts. We defer further discussion of P and C violations to a future paper.[14a]

If we knew all mass and coupling shifts, we could immediately test the prediction obtained above that the strong, electromagnetic, and (P- and C-conserving) weak symmetry violations all lie along a common

[13] As a minor complication, Coleman and Glashow (Ref. 1) have pointed out that if Cabibbo's assumption of a universal ratio of strangeness-changing to strangeness-conserving terms in the weak currents is relaxed, **10** and $\overline{\mathbf{10}}$ components can appear in the driving term.

[14] There are some possible exceptions. If more than one eigenvalue lies near one, the mass and coupling shifts will lie along a linear combination of the associated eigenvectors, and the linear combination may be different for the strong, electromagnetic, and weak symmetry violations. Thus the universality of the octet enhancement pattern would be somewhat reduced (see, however, Ref. 26). Another complication is that off-diagonal couplings such as the strangeness-changing weak decays require some special discussion; we defer this discussion to our future paper on the coupling shifts.

[14a] *Note added in proof.* P and C violations in the $BB\pi$ couplings have now been treated by R. Dashen, S. Frautschi, and D. Sharp, Phys. Rev. Letters **13**, 777 (1964).

eigenvector. The information actually available includes the strong mass splitting within the various established supermultiplets, isospin-violating electromagnetic mass splittings (available mainly in the B and Π supermultiplets which are stable against strong decays), nonleptonic weak couplings, and a few strong coupling shifts. At present, this information is sufficient to test the prediction of a common eigenvector only for the strong and electromagnetic mass shifts of B and Π. For this purpose, we consider the coefficients $\delta M_8{}^{B_s}$, $\delta M_8{}^{B_a}$, and[15] $(\delta M^2)_8{}^{\Pi}$ of normalized[16] octet mass matrices. Here, $\delta M_8{}^{B_s}$ and $\delta M_8{}^{B_a}$ refer to the symmetric and antisymmetric octet terms in the baryon mass. The data on strong splitting give

$$\delta M_8{}^{B_s}/\delta M_8{}^{B_a}\approx -0.24\,,\quad (\delta M^2)_8{}^{\Pi}/\bar{M}^{\Pi}\delta M_8{}^{B_a}\approx 2.9\,,\quad (2.7)$$

and the data on electromagnetic splitting give[17]

$$\begin{aligned}\delta M_8{}^{B_s}/\delta M_8{}^{B_a}&\approx -0.4\pm0.1\,,\\ (\delta M^2)_8{}^{\Pi}/\bar{M}^{\Pi}\delta M_8{}^{B_a}&\approx 2.7\pm0.7\,,\end{aligned}\quad (2.8)$$

in fair agreement with the prediction of a universal pattern[18] which applies if there is only one eigenvalue of A_8 near unity.

If the notion of a universal pattern of octet enhancement is accepted, it allows us to predict some of the mass and coupling shifts which are not yet known experimentally, without the labor of a detailed dynamical calculation. For example, octet dominance predicts equal spacing for all electromagnetic splittings within the Δ decuplet. The numerical value of this equally-spaced splitting can be obtained from the known electromagnetic splittings of the B octet together with the ratio

$$\delta M_8{}^{\Delta}/\delta M_8{}^{B_a}\approx 1.15 \quad (2.9)$$

which is observed for strong splittings and also holds for electromagnetic splittings if the enhanced eigenvector is universal. The prediction obtained in this way is[18,19]

$$M(N^{*-})-M(N^{*0})=2.8\text{ MeV}. \quad (2.10)$$

The phenomenology we have just been discussing refers to octet enhancement. To establish that the bootstrap mechanism is responsible for octet enhancement, detailed dynamical calculations are necessary. The remainder of the present paper is concerned with such calculations for the B octet and Δ decuplet.

Actually we will study only the A matrix, leaving the driving terms for future work.[20] This limited study already tells us a great deal, since the A matrix applies to all interactions and may provide the ratios among independent octet terms. The remaining questions which depend on the driving term, and which are consequently not discussed in the present paper, are: (i) the axis n in $SU(3)$ space along which the symmetry violation lies; (ii) the over-all scale of the violation (in spontaneous violation, which lacks a driving term, the scale must be set by higher order terms); (iii) there are some cases, such as the high accuracy of the Gell-Mann–Okubo mass formula in the baryon octet and the $|\Delta\mathbf{I}|=\frac{1}{2}$ rule for $K^+\to\pi^++\pi^0$ decay, where a large factor $(1-A_8)^{-1}$ may not be enough to provide the full explanation for octet dominance. The nature of the driving terms for weak and strong interactions is not very well understood, and it may be after all that these driving terms favor 8 over 27 violations. In our picture, octet dominance would then follow partly from the driving terms and partly from an eigenvalue of A_8 near unity, while the ratios among independent octet terms would still follow from the eigenvector associated with $A_8\approx1$.

III. CONVERGENCE OF THE DISPERSION RELATIONS FOR MASS AND COUPLING SHIFTS

In the accompanying paper[11] we have presented *exact* dispersion relations for first-order mass and coupling shifts. For practical purposes, of course, the dispersion integrals will be approximated by keeping only nearby singularities, and in order to justify this procedure we must show that the integrals converge well.

The dispersion relations which we use have the form[10]

$$\delta M=\frac{1}{R(D'(M))^2}\frac{1}{\pi}\left[\int_L\frac{D^2(W')\,\mathrm{Im}\delta T(W')dW'}{W'-M}+\int_R\frac{\mathrm{Im}[D^2(W')\delta T(W')]dW'}{W'-M}\right],\quad (3.1)$$

[15] We use M for fermions and M^2 for bosons because it is customary to do so, and because these are the combinations occurring naturally in the dispersion relations.

[16] That is, the trace of the square of the matrix is normalized to unity.

[17] To obtain the electromagnetic octet terms, we neglect the effects of strong symmetry breaking on the electromagnetic mass shifts. Some assumption of this sort is necessary because the $I=1$ components of the **27** and symmetric **8** matrices for $n=3$ are identical except for the difficult-to-observe, off-diagonal, term connecting Λ and Σ^0, or η and π^0. In practice, we calculate the amount of **27** from its $I=2$ component, which is the sole contributor to the deviation from equal spacing in I_3 in the Σ (or π) multiplet. Then the assumption that strong symmetry breaking has no effect enables us to deduce the $I=1$ component of the **27** from the $I=2$ component and thus to separate out the symmetric octet.

[18] The prediction that the electromagnetic mass shift should be characterized by the same ratios as the strong mass shifts is also made by the tadpole model (Ref. 1), and is in fact equivalent to the "hydbrid sum rules" of Coleman and Glashow.

[19] Significant corrections to this prediction, which result when one considers the effects of strong symmetry violations on the electromagnetic mass shifts, will be considered in Sec. VI.

[20] In the particular case of the neutron-proton mass difference, the driving terms have already been carefully studied, along with the A matrix, by R. Dashen, Phys. Rev. **135**, B1196 (1964).

$$\delta R = -\frac{RD''(M)\delta M}{D'(M)} + \frac{1}{(D'(M))^2}\frac{1}{2\pi i} \times \left[\int_L \frac{D^2(W')\,\mathrm{Im}\delta T(W')dW'}{(W'-M)^2} + \int_R \frac{\mathrm{Im}[D^2(W')\delta T(W')]dW'}{(W'-M)^2}\right], \quad (3.2)$$

Here, D is the denominator function for the strong interaction, R is the residue and M the position of the pole, and δT is the perturbation on the amplitude in which the pole appears. For the purpose of discussing convergence of the integrals appearing in our equations, it is sufficient to consider these single-channel versions of the general formalism. Also note that the integral written out in the coupling-shift relation converges more rapidly than the integral for δM, so it suffices to discuss the convergence of Eq. (3.1) for δM.

The amplitude T has the form

$$T = (e^{2i\eta}-1)/2i\rho\,, \quad (3.3)$$

where, in the πN and ΠB reactions considered in the present paper, ρ grows like W at large energies W. Thus on the right cut, unitarity places a bound of order $1/W$ on T and δT. We do not know how δT behaves far out on the left cut, but let us suppose it also falls off as fast as $1/W$ there.[21]

Similarly, we do not know the behavior of D at large W, but we can use potential theory as a guide. The D we are using is defined to have no poles or zeros on the physical sheet, except for the zeros at bound states. With this definition, it is shown in Appendix A that in potential theory D has the asymptotic behavior

$$D(q^2) \sim q^{2N} \quad \text{as} \quad q^2 \to \infty\,, \quad (3.4)$$

where N is the number of elementary particles in the channel under consideration.

Thus, if there are no elementary particles in the channel under consideration, $D(q^2)\sim 1$ at large q^2. Combining this behavior of D with the assumption that δT falls off at least as fast as $1/W$, we see that the integral in Eq. (3.1) converges quite well. If there were an elementary particle (the nucleon, say), then D would increase linearly at large energies and the integral in Eq. (3.1) would converge only slowly if at all. Thus the dispersion integrals in (3.1) and (3.2) may converge rapidly, which implies that reliable calculations can be done, but apparently we have to assume there are no elementary particles in the channel under consideration in order to obtain the good convergence.

In Dashen's calculation of the neutron-proton mass difference,[20] only the nearby singularities were included. The result agreed well with experiment. We interpret the success of Dashen's calculation optimistically, as indicating that the dispersion integral does converge rapidly and that the nucleon is not elementary.

In the present paper, we assume that good convergence does prevail in Eqs. (3.1) and (3.2). In line with this view, we make the following specific approximations: (i) We keep only low-mass states in each channel, ignoring, for example, the influence of higher ΠB resonances on B and Δ. (ii) In treating exchanges of the light-mass states in partial-wave amplitudes, we keep only the nearby singularities (i.e., the "short cuts" for B and Δ exchange).[22] (iii) Also in line with the hypothesis that B and Δ are composite particles, we shall use D functions which approach a constant as W approaches infinity.[23]

IV. SOME EFFECTS OF THE ASYMMETRY OF THE A MATRIX

The good convergence of the dispersion relations for mass and coupling shifts of composite particles, discussed in Sec. III, permits us to treat B and Δ in terms of a simple model where B and Δ poles occur in the ΠB scattering amplitude in response to B and Δ exchange. Shifts in the B and Δ parameters then depend only on δM^B, δM^Δ, δM^Π, $\delta\Gamma^{B\Delta\Pi}$, $\delta\Gamma^{BB\Pi}$, and driving terms. Also, static kinematics are approximately correct.

Under these circumstances, the effect of coupling changes on δM^B and δM^Δ is small, of order M^Π/M^B, as in the static model. Furthermore, the Π masses always appear in the combination[24] $(M^\Pi)^2$, with the result that δM^Π is always multiplied by M^Π and one finds that the effect of δM^Π on δM^B and δM^Δ is small, of order M^Π/M^B. The detailed derivation of these results is given in Appendix B.

Now suppose we calculate only the terms of the A matrix which connect δM^B and δM^Δ. This part of the A matrix by itself yields eigenvalues for A_1, A_8, and A_{27}. What happens to these eigenvalues when the terms of the A matrix involving coupling shifts, δM^Π, higher ΠB resonances and so forth are also considered? Can we infer from the result of Appendix B that the eigenvalues are only slightly modified? And can we see any signs that the universal pattern of octet enhancement, proposed in Sec. II, is emerging? The present section is devoted to these questions. The affirmative answers

[21] In Sec. II of R. Dashen and S. Frautschi, Phys. Rev. **135**, B1190 (1964), it was argued that the convergence of δT may in fact be considerably better than $1/W$.

[22] Through the dispersion integrals in (3.1) and (3.2) are assumed to converge when the complete δT is used, the contributions to δT from exchanges of individual particles with spins $J>1$ give divergent integrals as usual. Thus our first two approximations define our choice of cutoff for these terms.

[23] The reader may wonder about the convergence of the more approximate treatment of our letter (Ref. 8), where linear D functions were used. Actually, we also considered the B and Δ to be composite in the letter, assumed rapid convergence of the integrals, and therefore confined ourselves to low-mass states. Within the low-energy region, however, we approximated D by a linear function. The validity of this approximation for the particular case of N^* or Δ exchange is discussed in Appendix D of the present paper.

[24] This feature, which can be traced back to the quadratic form of the Klein-Gordon equation, was of course the original motivation for using M^2 in the sum rules for boson masses.

obtained encourage us to believe that the approximate results of the next section, where that part of the A matrix which connects δM^B and δM^Δ is studied and found to yield an eigenvalue near unity for A_8 but not for A_{27}, will remain essentially valid in fuller treatments of the A matrix.

First let us consider for a moment what would happen if the A matrix were symmetric. If the terms connecting δM^B to δM^Π, etc., were large the eigenvalues obtained by considering only δM^B and δM^Δ would not be reliable. On the other hand, if the terms connecting δM^B to δM^Π, etc., were small the previously obtained eigenvalues would be reliable but δM^Π, etc., would be essentially decoupled from δM^B and δM^Δ and could not exhibit octet enhancement unless A_8 had further eigenvalues near unity. In this case, the universality of the octet enhancement pattern, as discussed in Sec. II, would be somewhat reduced.

In fact, however, the A matrix is not symmetric. For example, in our model, B is not sensitive to shifts in the higher ΠB resonances, but the higher ΠB resonances are sensitive to shifts δM^B (see Appendix C for a more complete discussion). As a consequence, the gloomy prospect which a symmetric A matrix would have held out for us—that either our calculation of A_8 is unreliable or A_8 must have several eigenvalues near unity—may be avoided.

To illustrate what does happen, consider a hypothetical two by two A_8 matrix:

$$A_8=\begin{pmatrix} A_8^{11} & 0 \\ A_8^{21} & A_8^{22} \end{pmatrix}, \tag{4.1}$$

where $A_8^1\approx 1$. We may think of state 1 as that combination of B and Δ octet mass shifts which has an eigenvalue near one, and state 2 as a higher ΠB resonance which has no effect on B or Δ ($A_8^{12}\approx 0$) but is strongly influenced by B and Δ ($A_8^{21}\neq 0$). Now the eigenvalues of A_8 are the diagonal terms A_8^{11} and A_8^{22}, as one verifies by solving $\det(A_8-\lambda)=0$. Thus, state 2 does not change the eigenvalue $A_8^{11}\approx 1$ which was found by considering state 1 alone. The eigenvector[25] u^j associated with eigenvalue A_8^{11} in the two by two A matrix is the solution of $A_8^{ij}u^j=A_8^{11}u^i$:

$$u^2=(A_8^{12}/(A_8^{11}-A_8^{22}))u^1 \tag{4.2}$$

which, in general, may have a sizeable component of state 2. To see that it is actually this eigenvector that gets enhanced, consider the equation

$$(1-A_8)\begin{pmatrix}\delta M^1\\ \delta M^2\end{pmatrix}=\begin{pmatrix}D^1\\ D^2\end{pmatrix} \tag{4.3}$$

which has the solutions

$$\delta M^1=D^1/(1-A_8^{11}), \tag{4.4}$$

$$\delta M^2=(A_8^{12}/(1-A_8^{22}))\delta M^1+D^2/(1-A_8^{22}). \tag{4.5}$$

[25] Note that the eigenvectors of an asymmetric matrix need not be orthogonal.

If D^2 is of the same order as D^1, A_8^{22} is not near one, and A_8^{11} is near one, the solution (4.5) reduces to

$$\delta M^2\approx (A_8^{12}/(1-A_8^{22}))\delta M^1 \tag{4.6}$$

which essentially lies along the eigenvector (4.2).[26] These results are physically reasonable; since the higher ΠB resonance has little influence on B and Δ, it should not weaken the octet enhancement pattern they exhibit, but since B and Δ influence the higher resonance strongly their octet mass splitting tends to cause octet mass splitting in the higher resonance.

The real physical situation seems to resemble the hypothetical case we have just discussed, with more states participating of course. Suppose we start by considering just the influence of δM^B on δM^B. It turns out that this piece of the A matrix A^{BB} already has an eigenvalue A_8^{BB} near unity, whereas A_{27}^{BB} is far from unity (this fact is not displayed in the next section, where B and Δ are considered together throughout, but it can easily be derived from the material presented there). Next, consider δM^B and δM^Δ together. It is shown in the next section that $A^{\Delta B}$ is large whereas $A^{B\Delta}$ is relatively small (external mass effects turn out to give the largest numerical contributions to A, and there is no external Δ mass shift in our model to influence δM^B). Thus the eigenvalues obtained from A^{BB}, $A_8\approx 1$ and A_{27} far from **1**, are not greatly modified, but the eigenvector associated with $A_8\approx 1$ picks up a large δM^Δ component. Furthermore, $A^{\Delta\Delta}$ is small so the new eigenvalues introduced by including Δ are not close to unity.

A calculation has also been performed[27] incorporating δM^B, δM^Δ, and one higher resonance: the **35** which is conjectured to occur in the $\Pi\Delta$ channel.[28,29] Abers, Balázs, and Hara[28] have advanced a model of the **35** which has the sort of features we have been discussing: the **35** is influenced strongly by Δ and B but does not influence them in return. Dashen and Sharp[27] have extended the present formalism to this case and find, as one would expect, that A_8 still has an eigenvalue near unity, A_{27} still has no eigenvalue near unity, and the eigenvector associated with $A_8\approx 1$ picks up a large δM^{35} component.

We have argued in Appendix B that the effect of δM^Π, $\delta\Gamma^{B\Delta\Pi}$, and $\delta\Gamma^{BB\Pi}$ on δM^B and δM^Δ is rather small.

[26] It is interesting to note that if the second eigenvalue A_8^{22} also lies near one, δM^1 receives the usual enhancement, and the part of δM^2 lying along the eigenvector associated with A_8^{11} receives a double enhancement $(1-A_8^{22})^{-1}(1-A_8^{11})^{-1}$ according to Eqs. (4.4) and (4.5). Thus if A_8^{12} is not too small, the solution of (4.3) may still lie essentially along the eigenvector associated with A_8^{11} and a universal enhancement pattern may still occur. The double enhancement might produce unusually large shifts such as are needed to explain the large values observed for $\delta M^\Pi/M^\Pi$ and certain coupling constants.

[27] R. F. Dashen and D. H. Sharp (to be published). A similar calculation has been made by R. H. Capps, Phys. Rev. Letters **13**, 536 (1964).

[28] E. S. Abers, L. A. P. Balázs, and Y. Hara, Phys. Rev. **136**, B1382 (1964).

[29] G. Goldhaber, lecture at the Conference on Particle and High Energy Physics at Boulder, Colorado, 1964 (unpublished).

Thus, analogously to the higher ΠB and $\Pi\Delta$ resonances, there is reason to hope that coupling shifts and pion mass shifts will not greatly alter the eigenvalues we obtain by considering δM^B and δM^Δ. The converse problem of what influence δM^B and δM^Δ have on δM^Π, $\delta\Gamma^{B\Delta\Pi}$, and $\delta\Gamma^{BB\Pi}$, and its connection to the question of octet enhancement for these latter quantities, will not be considered here. We intend to return to the question of perturbations on the baryon couplings and pseudoscalar mesons, as well as vector mesons, in future publications.

V. CALCULATION OF THE A MATRIX FOR B AND Δ MASS SHIFTS

In the previous section, reasons have been given why a calculation of the A matrix elements connecting B and Δ mass shifts may be sufficient for obtaining an estimate of several physically interesting eigenvalues of A. In the present section, we proceed to calculate the elements of A_1 connecting the B and Δ mass shifts in part A; the corresponding elements of A_8, A_{27}, and A_{64} are obtained by group-theoretical methods in part B; the eigenvalues obtained for A are presented in part C; and the physically most interesting eigenvectors are given and compared with experiment in part D.

(A) We begin by reviewing briefly the $SU(3)$-symmetric reciprocal bootstrap model for B and Δ which, when combined with Eq. (3.1) for the mass shift, allows a calculation of the elements of A_1. In the reciprocal bootstrap model we consider pseudoscalar meson-baryon scattering, with B and Δ poles appearing in the direct channel and B and Δ exchanges in the crossed channel. Our approximations, as stated in Sec. III, involve keeping only the nearby "short cuts" from B and Δ exchange in the partial-wave amplitudes. In practice, the "short cuts" will be approximated by "pseudopoles."

We define our P-wave scattering amplitudes by the relation

$$T(W)=\frac{W^2}{(M^B)^2[(W-M^B)^2-(M^\Pi)^2]}\frac{e^{2i\eta}-1}{2iq}. \quad (5.1)$$

This choice of amplitude introduces no kinematic singularities in the W plane.[30,31] The $J=\frac{3}{2}^+$ amplitude has a direct channel pole which we approximate by $\gamma_{10}/(M^\Delta-W)$. In the $(\frac{1}{2})^+$ octet amplitudes, we have a coupled two-channel problem, because the octet representation occurs twice in the decomposition of $8\otimes 8$. Here the direct channel pole has the form $\mathbf{R}_8/(M^B-W)$, where in the usual symmetric-antisymmetric octet representation[32]

$$\mathbf{R}_8=\begin{pmatrix} 20/9 & (4\sqrt{5})\lambda/3 \\ (4\sqrt{5})\lambda/3 & 4\lambda^2 \end{pmatrix}\frac{\gamma_8}{(1+\lambda)^2}, \quad (5.2)$$

λ is the F/D ratio for the meson-baryon couplings,[32a] and $\gamma_8=3f_{\pi NN}^2/(M^\pi)^2$, $f_{\pi NN}^2\approx 0.08$.

Turning to the effect of B and Δ exchanges, one finds in the $(\frac{3}{2})^+$ decuplet amplitude a term $\gamma_{10}/12(W-2M^B+M^\Delta)$ from Δ exchange and a term

$$\frac{16(1+3\lambda)}{27(1+\lambda)^2}\frac{1}{(W-M^B)}\gamma_8$$

from B exchange. In the $(\frac{1}{2})^+$ octet amplitudes there is a term $\mathbf{R}_{10}{}^x/(W-2M^B+M^\Delta)$ from Δ exchange and a term $\mathbf{R}_8{}^x/(W-M^B)$ from B exchange, with the residues in the symmetric-antisymmetric octet representation given by[32]

$$\mathbf{R}_{10}{}^x=\begin{pmatrix} \frac{2}{3} & \frac{1}{3}\sqrt{5} \\ \frac{1}{3}\sqrt{5} & 0 \end{pmatrix}\gamma_{10}, \quad (5.3)$$

$$\mathbf{R}_8{}^x=\begin{pmatrix} (2/9)(3\lambda^2+1) & 0 \\ 0 & -(2/9)[3\lambda^2-(5/3)] \end{pmatrix}\frac{\gamma_8}{(1+\lambda)^2}. \quad (5.4)$$

In the dispersion relation (3.1) for δM^Δ, R is $-\gamma_{10}$ and normalizing $D'(M^\Delta)$ to unity, we obtain

$$\delta M^\Delta=-\frac{1}{2\pi i\gamma_{10}}\int\frac{D_{10}^2(W')\delta T_{10}(W')dW'}{W'-M^\Delta}, \quad (5.5)$$

where the contour of integration runs clockwise around the singularities and D_{10} and δT_{10} refer to the $(\frac{3}{2})^+$ decuplet channel.

Since the singlet mass shifts conserve $SU(3)$ symmetry, they can be obtained by varying the $SU(3)$-symmetric bootstrap. The contributions to δT_{10} from $SU(3)$-symmetric shifts in the exchanged Δ and B masses, respectively, are

$$\delta T_{10}\approx-\frac{\gamma_{10}}{12}\frac{\delta M^{\Delta\ \mathrm{exch}}}{(W-2M^B+M^\Delta)^2} \quad (5.6)$$

and[33]

$$\delta T_{10}\approx-\frac{16}{27}\frac{(1+3\lambda)}{(1+\lambda)^2}\gamma_8\frac{\delta M^{B\ \mathrm{exch}}}{(W-M^B)^2}. \quad (5.7)$$

Inserting (5.6) and (5.7) into the contour integral on

[30] W. Frazer and J. Fulco, Phys. Rev. **119**, 1420 (1960).
[31] S. Frautschi and J. Walecka, Phys. Rev. **120**, 1486 (1960).
[32] R. F. Dashen, Phys. Letters **11**, 89 (1964).

[32a] Here we refer to the F and D matrices introduced by Gell-Mann (Ref. 4). Note that these have the normalizations Tr $(D^2)=5/3$ and Tr $(F^2)=3$, whereas the matrices O we use to represent masses later in the present section have the normalization Tr $(O^2)=1$.

[33] There is a subtlety here. We are varying the exchanged mass $M^{B\ \mathrm{exch}}$, not the external mass $M^{B\ \mathrm{ext}}$. In passing from the pole of the complete amplitude at $u=(M^{B\ \mathrm{exch}})^2$ to its manifestation in the partial-wave amplitude, the external mass comes in and the pseudopole in the partial-wave amplitude is actually proportional to $1/(W-2M^{B\ \mathrm{ext}}+M^{B\ \mathrm{exch}})$. To obtain the sign of Eq. (5.7), we must vary only $M^{B\ \mathrm{exch}}$. After the variation is made, of course, $M^{B\ \mathrm{ext}}$ is lumped back together with $M^{B\ \mathrm{exch}}$ in the denominator.

the right side of (5.5), we obtain

$$\delta M^{\Delta}=-\frac{1}{12}\left(\frac{D_{10}{}^{2}(W')}{W'-M^{\Delta}}\right)'\Bigg|_{W'=2M^{B}-M^{\Delta}}\delta M^{\Delta\ \mathrm{exch}}$$

$$-\frac{16}{27}\frac{(1+3\lambda)}{(1+\lambda)^{2}}\frac{\gamma_{8}}{\gamma_{10}}\left(\frac{D_{10}{}^{2}(W')}{W'-M^{\Delta}}\right)'\Bigg|_{W'=M^{B}}\delta M^{B\ \mathrm{exch}}. \quad (5.8)$$

In view of the normalization $D_{10}'(M^{\Delta})=1$ and the fact that $D_{10}(M^{\Delta})=0$, the factor $(D^{2}/W'-M^{\Delta})'$ equals unity in the linear D approximation. In practice, the growth of D is expected to be less rapid so the factor will be somewhat smaller. We shall carry the factor along as a parameter, and find that the eigenvalues of A are not very sensitive to it for reasonable values of the parameter.

Now δM^{B} refers to any member of the octet. It is equal to $(8)^{-1/2}\delta M_{1}{}^{B}$, where $\delta M_{1}{}^{B}$ is the singlet mass shift and the coefficient $(8)^{-1/2}$ reflects our choice of normalization, explained in the next subsection. Similarly, we take $\delta M^{\Delta}=(10)^{-1/2}\delta M_{1}{}^{\Delta}$. Inserting these ratios into (5.8), and comparing Eq. (5.8) with (2.3), we can read off the exchange contributions to the A matrix:

$$A_{1}{}^{\Delta\Delta\ \mathrm{exch}}=-\frac{1}{12}\left(\frac{D_{10}{}^{2}(M^{x})}{M^{x}-M^{\Delta}}\right)', \quad (5.9)$$

$$A_{1}{}^{\Delta B\ \mathrm{exch}}=-\left(\sqrt{\frac{5}{4}}\right)\frac{16}{27}\frac{(1+3\lambda)}{(1+\lambda)^{2}}\frac{\gamma_{8}}{\gamma_{10}}\left(\frac{D_{10}{}^{2}(M^{B})}{M^{B}-M^{\Delta}}\right)', \quad (5.10)$$

where $M^{x}=2M^{B}-M^{\Delta}$.[34] The remaining mass shift contributions to $\delta M_{1}{}^{\Delta}$ come from the external B and Π masses. We have shown in Appendix B that the contribution of δM^{Π} is small, so we have only $A_{1}{}^{\Delta B\ \mathrm{ext}}$ to calculate. This is done by using the condition that the bootstrap theory is invariant under changes in the over-all mass scale.[7,11] According to this condition, the homogeneous equation

$$\frac{\delta M_{1}{}^{\Delta}}{M^{\Delta}}=A_{1}{}^{\Delta\Delta\ \mathrm{exch}}\frac{\delta M_{1}{}^{\Delta}}{M^{\Delta}}+A_{1}{}^{\Delta B\ \mathrm{exch}}\frac{\delta M_{1}{}^{B}}{M^{B}}+A_{1}{}^{\Delta B\ \mathrm{exch}}\frac{\delta M_{1}{}^{B}}{M^{B}}, \quad (5.11)$$

which is equivalent to

$$(\sqrt{10})\frac{\delta M^{\Delta}}{M^{\Delta}}=(\sqrt{10})A_{1}{}^{\Delta\Delta\ \mathrm{exch}}\frac{\delta M^{\Delta}}{M^{\Delta}}+(\sqrt{8})A_{1}{}^{\Delta B\ \mathrm{exch}}\frac{\delta M^{B}}{M^{B}}+(\sqrt{8})A_{1}{}^{\Delta B\ \mathrm{exch}}\frac{\delta M^{B}}{M^{B}}, \quad (5.12)$$

has a solution with $\delta M^{B}/M^{B}=\delta M^{\Delta}/M^{\Delta}$:

$$(5/4)^{1/2}=(5/4)^{1/2}A_{1}{}^{\Delta\Delta\ \mathrm{exch}}+A_{1}{}^{\Delta B\ \mathrm{exch}}+A_{1}{}^{\Delta B\ \mathrm{ext}}, \quad (5.13)$$

from which we find

$$A_{1}{}^{\Delta B\ \mathrm{ext}}=\left(\sqrt{\frac{5}{4}}\right)+\frac{1}{12}\left(\sqrt{\frac{5}{4}}\right)\left(\frac{D_{10}{}^{2}(M^{x})}{M^{x}-M^{\Delta}}\right)'+\frac{16}{27}\left(\sqrt{\frac{5}{4}}\right)\frac{(1+3\lambda)}{(1+\lambda)^{2}}\frac{\gamma_{8}}{\gamma_{10}}\left(\frac{D_{10}{}^{2}(M^{B})}{M^{B}-M^{\Delta}}\right)'. \quad (5.14)$$

To complete our tabulation of the A matrix, we of course have

$$A_{1}{}^{\Delta\Delta\ \mathrm{ext}}=0 \quad (5.15)$$

since Δ appears only in intermediate states in our model.

In order to make the corresponding calculations for the two-channel $(\frac{1}{2})^{+}$ octet problem, we begin with the matrix relation from our preceding paper[11]:

$$\delta M^{B}=-\frac{1}{2\pi i\,\mathrm{Tr}(\mathbf{R}_{8}\mathbf{R}_{8})}\times\mathrm{Tr}\left[\mathbf{R}_{8}\boldsymbol{\Delta}_{8}{}^{T}\int\frac{\mathbf{D}_{8}{}^{T}\delta\mathbf{T}_{8}\mathbf{D}_{8}}{W'-M^{B}}dW'\boldsymbol{\Delta}_{8}\right], \quad (5.16)$$

$$\boldsymbol{\Delta}=\lim_{W\to M^{B}}\left[(W-M^{B})\mathbf{D}^{-1}(W)\right], \quad (5.17)$$

where once more the contour of integration runs clockwise around the singularities, and the matrices $\mathbf{D}_{8}$ and $\delta\mathbf{T}_{8}$ refer to the $(\frac{1}{2})^{+}$ octet channels.

As discussed in the concluding portion of our previous paper,[11] it is reasonable to assume that the D/F ratio does not depend strongly on energy. In this case, when we diagonalize D at one energy (e.g., the energy of the Δ exchange pole) it remains approximately diagonal over the whole low-energy region. A further simplification is to give the diagonalized D function a linear energy dependence

$$\mathbf{D}_{8}(W)=\mathbf{D}_{8}(M^{B})+\mathbf{1}(W-M^{B}),$$

with $\mathbf{1}$ the unit matrix. Recalling that $\mathbf{D}_{8}(M^{B})$ vanishes in the channel which has the bound state (with $R_{8}\neq 0$), we see that Eq. (5.16) reduces to[34a]

$$\delta M^{B}=-\frac{1}{2\pi i\,\mathrm{Tr}(\mathbf{R}_{8}\mathbf{R}_{8})}\int(W'-M^{B})\times\mathrm{Tr}[\mathbf{R}_{8}\delta\mathbf{T}_{8}(W')]dW'. \quad (5.18)$$

There are two kinds of correction to this approximate expression. In the first place, the coefficient of the unit

[34] There is also a factor $\bar{M}^{B}/\bar{M}^{\Delta}\approx 1150/1385$ in $A_{1}{}^{B\Delta\ \mathrm{exch}}$, which we set equal to unity since we are generally ignoring terms of order $(M^{\Delta}-M^{B})/M^{B}$ and M^{Π}/M^{B}.

[34a] The reduction of (5.16) to (5.18) is aided by conditions such as $\mathbf{D}(M^{B})\mathbf{R}=0$, which we obtain by recalling that the residue $\mathbf{R}_{8}$ factors into a product of couplings $f_{i}f_{j}$, and that f_{i} is an eigenvector associated with the eigenvalue zero of $\mathbf{D}$. Then the assumption that $\mathbf{D}$ is approximately diagonal in the neighborhood of the pole, together with our normalization for $\mathbf{D}$, leads to the condition $\boldsymbol{\Delta}\mathbf{R}=\mathbf{R}$.

matrix may contain curvature. We can make reasonable estimates of the curvature, and for that purpose will introduce a scalar factor $[D_8{}^2/(W-M^B)]'$ into the result of integrating Eq. (5.18), completely analogously to the factor that appeared in the expressions for δM^Δ. In the second place, as discussed at the end of our previous paper,[11] $\mathbf{D}_8$ may introduce off-diagonal elements. Since a single term, Δ exchange, dominates the left cut in our model, however, one estimates[35] that these off-diagonal elements are small, and we shall assume they do not build up to any appreciable extent over the rather short distance from $W=M^B$ to the Δ exchange pseudopolar at $W=2M^B-M^\Delta$.

We can now proceed to evaluate Eq. (5.18). The left-hand singularities due to singlet shifts in exchanged masses are[33]

$$\delta\mathbf{T}_8\approx-\frac{\mathbf{R}_{10}{}^x\delta M^{\Delta\ \mathrm{exch}}}{(W-2M^B+M^\Delta)^2}-\frac{\mathbf{R}_8{}^x\delta M^{B\ \mathrm{exch}}}{(W-M^B)^2}. \quad (5.19)$$

Performing the traces and integration in Eq. (5.18), relating δM's to δM_1's, and extracting the coefficients of $\delta M_1{}^{\Delta\ \mathrm{exch}}$ and $\delta M_1{}^{B\ \mathrm{exch}}$, we obtain

$$A_1{}^{B\Delta\ \mathrm{exch}}=-\left(\sqrt{\frac{4}{5}}\right)\frac{15(1+\lambda)^2(1+3\lambda)}{2(5+9\lambda^2)^2}\frac{\gamma_{10}}{\gamma_8}\times\left(\frac{D_8{}^2(M^x)}{M^x-M^B}\right)', \quad (5.20)$$

$$A_1{}^{BB\ \mathrm{exch}}=-\frac{[5+30\lambda^2-27\lambda^4]}{2(5+9\lambda^2)^2}, \quad (5.21)$$

(the $[D^2/(W-M^B)]'$ factor is precisely one at the B exchange pole). As before, $A_1{}^{BB\ \mathrm{ext}}$ is given by the mass scale invariance of the theory, and is

$$A_1{}^{BB\ \mathrm{ext}}=1+\frac{[5+30\lambda^2-27\lambda^4]}{2(5+9\lambda^2)^2}+\frac{15(1+\lambda)^2(1+3\lambda)}{2(5+9\lambda^2)^2}\frac{\gamma_{10}}{\gamma_8}\left(\frac{D_8{}^2(M^x)}{M^x-M^B}\right)', \quad (5.22)$$

while, of course,

$$A_1{}^{B\Delta\ \mathrm{ext}}=0. \quad (5.23)$$

We have now found all contributions to the A coefficients relating $\delta M_1{}^B$ and $\delta M_1{}^\Delta$. The contributions depend only on λ, γ_8/γ_{10}, and the curvature of the D functions. For λ, several arguments[36,37] indicate a value in the range $0.3\lesssim\lambda\lesssim0.5$ (in agreement with the result of the reciprocal bootstrap[32,35]; we shall carry λ along as a parameter through the calculation of the A matrix and let it vary over the range 0.3 to 0.5 at the end of the calculation. For γ_8/γ_{10}, we use the relation

$$\gamma_{10}=\frac{64}{99}\frac{(1+3\lambda)}{(1+\lambda)^2}\gamma_8 \quad (5.24)$$

from the reciprocal bootstrap theory [alternatively, we could have used the rather poorly defined experimental result $\gamma_8/\gamma_{10}\approx\frac{3}{2}$, which agrees with (5.24) over the range of acceptable λ]. Inserting (5.24) into the expressions for the A coefficients, we finally obtain A_1 values which depend only on λ and the curvature of D. These values have been collected together in Tables I and II.

TABLE I. Contributions to A_1 from shifts in exchanged masses.

	B exch	Δ exch
B	$-\dfrac{[5+30\lambda^2-27\lambda^4]}{2(5+9\lambda^2)^2}$	$-\dfrac{160}{33}\left(\sqrt{\dfrac{4}{5}}\right)\dfrac{(1+3\lambda)^2}{(5+9\lambda^2)^2}\left(\dfrac{D_8{}^2(M^x)}{M^x-M^B}\right)'$
Δ	$-\dfrac{11}{12}\left(\sqrt{\dfrac{5}{4}}\right)\left(\dfrac{D_{10}{}^2(M^B)}{M^B-M^\Delta}\right)'$	$-\dfrac{1}{12}\left(\dfrac{D_{10}{}^2(M^x)}{M^x-M^\Delta}\right)'$

(B) We now turn from singlet mass shifts to $SU(3)$ symmetry-violating shifts. The representations which appear in the B and Δ mass matrices are just the irreducible representations appearing in $\mathbf{8}\otimes\mathbf{8}=\mathbf{1}+\mathbf{8}+\mathbf{8}+\mathbf{10}+\overline{\mathbf{10}}+\mathbf{27}$ and in $\mathbf{10}\otimes\mathbf{10}=\mathbf{1}+\mathbf{8}+\mathbf{27}+\mathbf{64}$. For a discussion of electromagnetic mass shifts to order e^2, we need only **1**, **8**, and **27** terms since the electric current transforms like **8** and the e^2 terms involve a symmetric current-current product. Similarly, only **1**, **8**, and **27** mass perturbations appear to second order in Ne'eman's fifth interaction (the presence of a singlet term in the current obviously does not change this conclusion). In a study of the possibility of spontaneous $SU(3)$ violation, on the other hand, all possible representations should be considered. We shall actually consider **1**, **8**, **27**, and **64** but not **10** and $\overline{\mathbf{10}}$ (in addition to being absent in electromagnetic corrections, the **10** and $\overline{\mathbf{10}}$ representations have no $Y=0$, $I=0$ component and therefore do not appear in hypercharge- and isospin-conserving strong interactions).

The elements of A_8, A_{27}, and A_{64} can be obtained from A_1 by group-theoretical techniques without further explicit reference to the dispersion integrals. The appropriate techniques were developed by Cutkosky and Tarjanne,[7] and we have discussed in detail how to carry them through in our previous paper.[11] One first

TABLE II. Contributions to A_1 from shifts in external masses.

	B ext	Δ ext
B	$1-A_1{}^{BB\ \mathrm{exch}}-(5/4)^{1/2}A_1{}^{B\Delta\ \mathrm{exch}}$	0
Δ	$(5/4)^{1/2}-A_1{}^{\Delta B\ \mathrm{exch}}-(5/4)^{1/2}A_1{}^{\Delta\Delta\ \mathrm{exch}}$	0

[35] Y. Hara, Phys. Rev. **135**, B1079 (1964).
[36] A. Martin and K. Wali, Phys. Rev. **130**, 2455 (1963).
[37] R. Dalitz, Phys. Letters **5**, 53 (1963).

TABLE III. Diagonal elements of mass matrices for the $J=\frac{3}{2}^+$ decuplet.

Representation	N^*	Y^*	Ξ^*	Ω	Normalization factor
1	1	1	1	1	$\sqrt{10}$
8	1	0	-1	-2	$\sqrt{10}$
27	-3	5	3	-9	$\sqrt{210}$
64	1	-4	6	-4	$\sqrt{140}$

expands an arbitrary mass shift into a sum over irreducible representations, i.e.,

$$\delta M^{\Delta}=O_1\delta M_1{}^{\Delta}+\sum_{n=1}^{8}O_{8,n}\delta M_{8,n}{}^{\Delta}+\sum_{n=1}^{27}O_{27,n}\delta M_{27,n}{}^{\Delta}+\sum_{n=1}^{64}O_{64,n}\delta M_{64,n}{}^{\Delta}, \quad (5.25)$$

and similarly for δM^B. Here the O's are matrices which have the indicated transformation properties, have dimensions 10×10 for the Δ and 8×8 for the B, and are normalized according to the condition $\mathrm{Tr}(O^2)=1$. Since the elements of A are independent of n, we need only consider a single n for each irreducible representation. It is most convenient to choose n such that O is diagonal in the usual basis set of the physical particles, where hypercharge and isotopic spin are diagonal (e.g., for Δ, the matrix $O_{8,n=8}$ is diagonal in the basis set N^*, Y_1^*, Ξ^*, Ω). We can then tabulate the O matrices that will be used by exhibiting their diagonal elements, and this is done in Tables III and IV. In these tables all diagonal elements within a given isotopic spin multiplet are the same, and all must be summed over to arrive at the normalization factor, in the last column, by which each matrix is divided. For example, in Table III, O_1 is a 10-by-10 unit matrix with each term divided by $\sqrt{10}$. Note that the different O matrices for each supermultiplet are orthogonal, as they should be. The octet entry in the Δ table and the $\mathbf{8}_a$ entry in the B table will be recognized as the hypercharge, and the $\mathbf{8}_s$ entry in the B table as just the D-type coupling coefficients of $B\bar{B}$ to the isosinglet η.

The group theoretical calculation for ratios of elements of A that involve external mass shifts is quite simple.[7,11] Consider, for example, the effect of changes in external baryon mass on δM^{Δ}. It is easiest to study δM^{Ω} because Ω appears only in the $\Xi\bar{K}$ channel. We can think of Ω as being in the $I=0$ state

$$|\Omega^-\rangle=|\Xi^0K^-\rangle+|\Xi^-K^0\rangle/\sqrt{2}, \quad (5.26)$$

that is, with 50% probability of appearing in the Ξ^0K^- channel and 50% probability of appearing in the Ξ^-K^0 channel. The change in Ω mass, when M^K is held fixed and the external Ξ mass is varied, is then

$$\delta M^{\Omega}=c(\tfrac{1}{2}\delta M^{\Xi^0\ \mathrm{ext}}+\tfrac{1}{2}\delta M^{\Xi^-\ \mathrm{ext}}), \quad (5.27)$$

where c is a number which must be determined by dynamical considerations such as we have used in computing $A_1{}^{\Delta B\ \mathrm{ext}}$. Now suppose the mass shifts are pure **1**. According to Eq. (5.25), we have

$$\delta M^{\Omega}=O_1\delta M_1{}^{\Delta}, \quad (5.28)$$

and a similar relation for the δM^{Ξ}'s. We take the values of O_1 from Tables III and IV

$$\delta M^{\Omega}=(1/\sqrt{10})\delta M_1{}^{\Delta}, \quad (5.29)$$

$$\delta M^{\Xi^0}=\delta M^{\Xi^-}=(1/\sqrt{8})\delta M_1{}^{B}, \quad (5.30)$$

and inserting (5.29) and (5.30) into (5.27), we find

$$\delta M_1{}^{\Delta}=(5/4)^{1/2}c\delta M_1{}^{B\ \mathrm{ext}}, \quad (5.31)$$

which implies that

$$A_1{}^{\Delta B\ \mathrm{ext}}=(5/4)^{1/2}c. \quad (5.32)$$

Next, suppose the mass shifts are pure **8**. In that case, we have relations such as

$$\delta M^{\Omega}=O_{8,n}\delta M_{8,n}{}^{\Delta} \quad (5.33)$$

and taking the values of $O_{8,8}$ from Tables III and IV, we find

$$\delta M^{\Omega}=-(2/\sqrt{10})\delta M_{8,8}{}^{\Delta}, \quad (5.34)$$

$$\delta M^{\Xi^0}=\delta M^{\Xi^-}=-\tfrac{1}{2}\delta M_{8,8}{}^{B_a}-(1/2\sqrt{5})\delta M_{8,8}{}^{B_s}. \quad (5.35)$$

Inserting (5.34) and (5.35) into (5.27), we find

$$\delta M_{8,8}{}^{\Delta}=\tfrac{1}{4}(\sqrt{10})c\delta M_{8,8}{}^{B_a\ \mathrm{ext}}+\tfrac{1}{4}\sqrt{2}c\delta M_{8,8}{}^{B_s\ \mathrm{ext}}, \quad (5.36)$$

which implies that

$$A_8{}^{\Delta B_a\ \mathrm{ext}}=\tfrac{1}{4}(\sqrt{10})c, \quad (5.37)$$

$$A_8{}^{\Delta B_s\ \mathrm{ext}}=\tfrac{1}{4}\sqrt{2}c. \quad (5.38)$$

Finally, comparing (5.38) with (5.32), we see that

$$A_8{}^{\Delta B_a\ \mathrm{ext}}/A_1{}^{\Delta B\ \mathrm{ext}}=1/\sqrt{2}, \quad (5.39)$$

$$A_8{}^{\Delta B_s\ \mathrm{ext}}/A_1{}^{\Delta B\ \mathrm{ext}}=1/\sqrt{10}. \quad (5.40)$$

Proceeding in this way, we construct Table V for the ratios among various $A^{B\Delta\ \mathrm{ext}}$, and Table VI for the ratios among $A^{BB\ \mathrm{ext}}$.

Several features which occur in Tables V and VI, and which will also apply to the elements of A connecting exchanged mass shifts, are worth mentioning. In the first place, all elements of A connecting different representations such as **1** and **8** are zero, as we discussed

TABLE IV. Diagonal elements of mass matrices for the $J=\frac{1}{2}^+$ octet.

Representation	N	Λ	Σ	Ξ	Normalization factor
1	1	1	1	1	$\sqrt{8}$
8_a	1	0	0	-1	2
8_s	-1	-2	2	-1	$2\sqrt{5}$
27	-3	9	1	-3	$2\sqrt{30}$

in Sec. II. Secondly, Table VI is symmetric between 8_a and 8_s, as it must be on the general grounds discussed in Appendix C. Thirdly, the entries in Table VI depend only on even powers of the F/D ratio λ, except for the terms connecting 8_s to 8_a which depend only on odd powers of λ. This pattern can be explained in terms of Gell-Mann's reflection operator R.[4] Under R, baryon mass shifts transforming like **1**, $\mathbf{8}_s$, and **27** are unchanged, whereas shifts transforming like $\mathbf{8}_a$ undergo a sign change. In order for relations like $\delta M_1{}^B = A_1{}^{BB}\delta M_1{}^B$ to continue to hold after application of R reflection,

TABLE V. Ratios between $A_r{}^{\Delta B \text{ ext}}$ and $A_1{}^{\Delta B \text{ ext}}$.

Representation of δM^Δ	Representation of $\delta M^{B \text{ ext}}$ 1	8_a	8_s	27
1	1	0	0	0
8	0	$1/\sqrt{2}$	$1/\sqrt{10}$	0
27	0	0	0	$(\sqrt{7})/3\sqrt{5}$

$A_1{}^{BB}$, $A_8{}^{B_sB_s}$, $A_8{}^{B_aB_a}$, and $A_{27}{}^{BB}$ must be even under R while $A_8{}^{B_sB_a}$ must be odd. Since F and D couplings are, respectively, odd and even under R, the desired behavior is obtained if $A_8{}^{B_sB_a}$ is odd in λ (each power of λ represents one F-type coupling) whereas $A_1{}^{BB}$, etc., are even in λ.

Next we turn to the group theoretical calculation for ratios of elements of A that involve shifts in exchanged masses.[7,11] Consider, for example, the effect of changes in exchanged Δ mass on δM^Δ. Again, it is easiest to study δM^Ω, with Ω appearing only in the reaction $\Xi\bar{K} \to \Xi\bar{K}$.

TABLE VI. Ratios between $A_r{}^{BB \text{ ext}}$ and $A_1{}^{BB \text{ ext}}$.

Representation of δM^B	Representation of $\delta M^{B \text{ ext}}$ 1	8_a	8_s	27
1	1	0	0	0
8_a	0	$\frac{1}{2}$	$-\frac{(3\sqrt{5})\lambda}{5+9\lambda^2}$	0
8_s	0	$-\frac{(3\sqrt{5})\lambda}{5+9\lambda^2}$	$\frac{-3+9\lambda^2}{10+18\lambda^2}$	0
27	0	0	0	$\frac{1-3\lambda^2}{5+9\lambda^2}$

We are interested in the effect of mass shifts of the Δ pole in the crossed reaction. Since the crossed reaction is $\Xi K \to \Xi K$, the Δ pole in question is the $Y_1{}^*$. Thus, holding M^K fixed, we have[38]

$$\delta M^\Omega = c'\delta M^{Y_1{}^* \text{ exch}}, \tag{5.41}$$

where c' is a number that must be determined dynamically. From Table III, the singlet shifts are

$$\delta M^\Omega = \delta M^{Y_1{}^*} = (1/\sqrt{10})\delta M_1{}^\Delta \tag{5.42}$$

from which we deduce

$$\delta M_1{}^\Delta = c'\delta M_1{}^{\Delta \text{ exch}} \tag{5.43}$$

and

$$A_1{}^{\Delta\Delta \text{ exch}} = c'. \tag{5.44}$$

Similarly, the octet shifts are

$$\delta M_{8,8}{}^\Omega = -(2/\sqrt{10})\delta M_{8,8}{}^\Delta, \tag{5.45}$$

$$\delta M_{8,8}{}^{Y_1{}^*} = 0 \tag{5.46}$$

from which we deduce

$$A_B{}^{\Delta\Delta \text{ exch}} = 0, \tag{5.47}$$

and the **27** shifts are

$$\delta M_{27}{}^\Omega = -[9/\sqrt{(210)}]\delta M_{27}{}^\Delta = -(9/5)\delta M_{27}{}^{Y_1{}^*}, \tag{5.48}$$

from which we deduce that

$$\delta M_{27,n}{}^\Delta = -(5/9)c'\delta M_{27,n}{}^{\Delta \text{ exch}}, \tag{5.49}$$

$$A_{27}{}^{\Delta\Delta \text{ exch}} = -(5/9)c', \tag{5.50}$$

and

$$A_{27}{}^{\Delta\Delta \text{ exch}}/A_1{}^{\Delta\Delta \text{ exch}} = -5/9. \tag{5.51}$$

Proceeding in this way, we construct Tables VII–X for the ratios among the various elements of $A^{\Delta\Delta \text{ exch}}$, $A^{B\Delta \text{ exch}}$, $A^{\Delta B \text{ exch}}$, and $A^{BB \text{ exch}}$. It will be noted, in accordance with Appendix C, that the matrix expressing ratios among the elements of $A^{B\Delta \text{ exch}}$ is just the transpose of the corresponding matrix for $A^{\Delta B \text{ exch}}$.

(C) We have now tabulated information which allows us to calculate all elements of the A matrix connecting

TABLE VII. Ratios between $A_r{}^{\Delta\Delta \text{ exch}}$ and $A_1{}^{\Delta\Delta \text{ exch}}$.

Representation of δM^Δ	Representation of $\delta M^{\Delta \text{ exch}}$ 1	8	27	64
1	1	0	0	0
8	0	0	0	0
27	0	0	$-5/9$	0
64	0	0	0	1

TABLE VIII. Ratios between $A_r{}^{B\Delta \text{ exch}}$ and $A_1{}^{B\Delta \text{ exch}}$.

Represention of δM^B	Representation of $\delta M^{\Delta \text{ exch}}$ 1	8	27
1	1	0	0
8_a	0	0	0
8_s	0	$-\frac{1}{4}\left(\sqrt{\frac{2}{5}}\right)\left[\frac{5+6\lambda+9\lambda^2}{1+3\lambda}\right]$	0
27	0	0	$-\left(\sqrt{\frac{7}{5}}\right)\left[\frac{\lambda(1-\lambda)}{1+3\lambda}\right]$

[38] We shall not refer explicitly to the charge states this time, and we could also have discussed external mass shifts without referring to them, because the mass matrices of Table IV do not distinguish between members of a given multiplet.

TABLE IX. Ratios between $A_r^{\Delta B\,\mathrm{exch}}$ and $A_1^{\Delta B\,\mathrm{exch}}$.

Representation of δM^Δ	Representation of $\delta M^{B\,\mathrm{exch}}$ 1	8_a	8_s	27
1	1	0	0	0
8	0	0	$-\frac{1}{4}\left(\sqrt{\frac{2}{5}}\right)\left[\frac{5+6\lambda+9\lambda^2}{1+3\lambda}\right]$	0
27	0	0	0	$-\left(\sqrt{\frac{7}{5}}\right)\left[\frac{\lambda(1-\lambda)}{1+3\lambda}\right]$

δM^B and δM^Δ. To calculate $A_{27}{}^{\Delta B}$, for example, we write

$$A_{27}{}^{\Delta B}=\left(\frac{A_{27}{}^{\Delta B\,\mathrm{exch}}}{A_1{}^{\Delta B\,\mathrm{exch}}}\right)A_1{}^{\Delta B\,\mathrm{exch}}+\left(\frac{A_{27}{}^{\Delta B\,\mathrm{ext}}}{A_1{}^{\Delta B\,\mathrm{ext}}}\right)A_1{}^{\Delta B\,\mathrm{ext}}, \quad (5.52)$$

and look up all the quantities on the right side of the equation in the appropriate tables.

The elements of A depend on λ and on the curvature of the strong interaction D functions. We have calculated A with λ varying over the range $0.3\leq\lambda\leq 0.5$ suggested by several experimental and theoretical arguments.[32,35–37] The D functions were given the form

$$D=(W-M)(M-M')/(W-M'), \quad (5.53)$$

where $M'>M$. This form for D has the desirable features that D approaches a constant as W approaches ∞, and the singularity of D is on the right cut. M' was varied over the reasonable range $M'=2M$ to $M'=\infty$. The results were not particularly sensitive to these variations.

It is found that the external mass terms in A are generally larger than the exchanged mass terms. This may help explain the good results of some earlier approximate calculations which have been made including only estimates of external mass terms.[9]

To illustrate the kind of results that are obtained, let us set $M'=\infty$ so that all factors $[D^2(W)/(W-M)]'$ simplify to unity. (This was the case considered in our letter.[8]) At $\lambda=0.46$, for example, the A matrices have the numerical values

$$A_1=\begin{pmatrix}1.58 & -0.52\\ 1.20 & -0.08\end{pmatrix} \quad (5.54)$$

and

$$A_{27}=\begin{pmatrix}0.23 & 0.06\\ 1.00 & 0.05\end{pmatrix}, \quad (5.55)$$

where the first row and column refer to B and the second to Δ,

$$A_8=\begin{bmatrix}0.84 & -0.66 & 0\\ -0.66 & 0.04 & 0.30\\ 1.57 & 1.32 & 0\end{bmatrix}, \quad (5.56)$$

where the first row and column refer to the antisymmetric octet of baryon masses, the second to the symmetric octet of baryon masses, and the third to Δ, and

$$A_{64}{}^{\Delta\Delta}=-0.08. \quad (5.57)$$

The eigenvalues of A are obtained from the condition $\det(A-\lambda)=0$. Applying this condition to matrices (5.54)–(5.57), we find the eigenvalues $A_1=1.00$ and 0.54, $A_8=1.00$ and 0.72 and -0.93, $A_{27}=0.41$ and -0.13, and $A_{64}=-0.08$.

The unit eigenvalue of A_1 is obtained for all λ, and its associated eigenvector always has equal B and Δ mass shifts. This just expresses the invariance of the bootstrap theory under simultaneous changes of all masses by the same scale factor, which we have explicitly imposed on the A matrix.

The other eigenvalue of A_1, and the eigenvalues of A_{27} and A_{64}, are far from unity, indicating that the strong interactions are stable against displacements of these types and that singlet and 27-plet electromagnetic mass corrections are not enhanced. The unit eigenvalue for A_8 indicates that octet electromagnetic corrections are enhanced, and the $SU(3)$-symmetric bootstrap equations for the strong interactions are possibly unstable. No particular significance should be attached to the result that the octet eigenvalue is exactly one; the eigenvalue varies from about 0.9 to 1.1 over the range of λ considered and we have simply presented the results at the point where it crosses unity. It may be mentioned that although we have not considered A_{10}, for reasons mentioned earlier in this section, the specific element $A_{10}{}^{BB}$ was calculated at one stage of our investigation. It was far from one, whereas A_8 already has an eigenvalue near one when only the baryon mass terms are included.

Actually, we should have given the D function some curvature in calculating the A matrix. A reasonable upper limit for the curvature would be obtained by setting M' in (5.53) equal to $2M$. Using this D function gives qualitatively the same results as a straight-line D function. For example, with $\lambda=0.33$, we obtain

TABLE X. Ratios between $A_r^{BB\,\mathrm{exch}}$ and $A_1^{BB\,\mathrm{exch}}$.

Representation of δM^B	Representation of $\delta M^{B\,\mathrm{exch}}$ 1	8_a	8_s	27
1	1	0	0	0
8_a	0	0	$-\frac{(8\sqrt{5})\lambda}{5+30\lambda^2-27\lambda^4}$	0
8_s	0	$-\frac{(8\sqrt{5})\lambda}{5+30\lambda^2-27\lambda^4}$	$-\frac{8(1+9\lambda^2)}{3(5+30\lambda^2-27\lambda^4)}$	0
27	0	0	0	$-\frac{13+18\lambda^2-45\lambda^4}{3(5+30\lambda^2-27\lambda^4)}$

eigenvalues $A_1=1.0$ and ≈ 0.2, $A_8\approx 0.9$ and 0.2 and -0.6, $A_{27}\approx 0.2$ and 0.0, and $A_{64}\approx 0.0$.

It would be interesting if a general reason for $A_8\approx 1$ could be produced without going through the sort of detailed calculation we have performed. The fact that the baryon mass shift contains two independent octet matrices helps by providing A_8 with one more eigenvalue than A_1 or A_{27}, thus improving the chance that an eigenvalue of A_8 lies near one, but of course this is hardly an explanation.

(D) The eigenvector with which we are particularly concerned is the enhanced octet eigenvector associated with the eigenvalue lying near one. This eigenvector depends somewhat on λ and the curvature of D; for $\lambda=0.46$ and linear D, it is

$$\delta M_8^{B_s}/\delta M_8^{B_a}\approx -0.24\,,\quad \delta M_8^{\Delta}/\delta M_8^{B_a}\approx 1.30\,,\qquad(5.58)$$

where the δM's are defined by Eq. (5.25). For $\lambda=0.33$ and $M'=2M$ (strongly curved D), the eigenvector is

$$\delta M_8^{B_s}/\delta M_8^{B_a}\approx -0.5\,,\quad \delta M_8^{\Delta}/\delta M_8^{B_a}\approx 0.8\,,\qquad(5.59)$$

and for intermediate values of λ and M', the eigenvector generally lies between these extremes. The corresponding experimental ratios, as discussed in Sec. II, are

$$\delta M_8^{B_s}/\delta M_8^{B_a}\approx -0.25\,,\quad \delta M_8^{\Delta}/\delta M_8^{B_a}\approx 1.25\qquad(5.60)$$

for the strong mass splitting ($n=8$), and

$$\delta M_8^{B_s}/\delta M_8^{B_a}\approx -0.4\pm 0.1\qquad(5.61)$$

for the electromagnetic mass splitting ($n=3$). Thus, experiment supports the prediction that both strong and electromagnetic mass shifts lie along the eigenvector of A_8 whose eigenvalue is close to one.

The second octet eigenvalue is also of some interest for the case $M'=\infty$, $\lambda=0.46$, since it lies rather near one. Its eigenvector is

$$\delta M_8^{B_s}/\delta M_8^{B_a}\approx 0.18\,,\quad \delta M_8^{\Delta}/\delta M_8^{B_a}\approx 2.60\,.\qquad(5.62)$$

Note that the first two octet eigenvectors are not orthogonal; this is to be expected because the A matrix is not symmetric. Actually the two eigenvectors make a rather small angle with respect to one another, which ensures that even if the second one should contribute substantially, the ratios of the mass matrices built up by the enhancement would not change too much.

VI. HIGHER ORDER CORRECTIONS

The theory we have presented thus far only takes first-order violations of $SU(3)$ symmetry into account. There are also higher order violations, of course. Usually these are not large enough to change the qualitative features of a first-order calculation, but in certain cases the strong symmetry violation may act more than once, or in conjunction with an electromagnetic or weak symmetry violation, to produce an effect quite different than the first-order result.

Higher order processes will affect our formalism in several different ways. In the first place, there will be second-order driving terms which include new representations of $SU(3)$. In the second place, the A matrix will acquire a noninvariant part, transforming like 8 and 27. The A matrix now connects parts of the mass shift belonging to different irreducible representations of $SU(3)$, such as δM_8 and δM_{27}. Since the strong symmetry-breaking terms conserve isotopic spin, however, the A matrix still connects only parts of the mass shift belonging to the same irreducible representations of $SU(2)$. The eigenvalues of A, formerly associated with irreducible representations of $SU(3)$, each split into several eigenvalues associated with irreducible representations of $SU(2)$. Yet another effect of second-order corrections is to introduce new terms such as $(\delta M)^2$. This is not so bad for terms such as $\delta M_{\text{strong}}\,\delta M_{\text{em}}$ which occur when one includes strong violations in the study of electromagnetic and weak corrections. In this case, the strong mass and coupling shifts can be taken as given by the solution of the strong symmetry-violation problem, and incorporated into the noninvariant part of the A matrix, leaving us with a linear equation for the weak or electromagnetic shifts. The second-order strong symmetry violations are intrinsically more difficult to treat.

Obviously, then, higher order corrections introduce considerable complication. As often happens in physics, however, when the higher order corrections become very strong and must be taken into account, new ways to reduce the complexity of the problem suggest themselves. The particular simplification we have in mind here is that the very violence of the symmetry breaking sometimes reduces the calculation to a relatively simple $SU(2)$ problem.

One such case is the electromagnetic mass splittings of the $J=\frac{3}{2}^+$ resonance N_3^*.[39] In $SU(3)$ this resonance is in the state

$$|N^*\rangle=1/\sqrt{2}\,|\pi N\rangle+1/\sqrt{2}\,|K\Sigma\rangle.\qquad(6.1)$$

Strong symmetry breaking drives the πN threshold far below the $K\Sigma$ threshold, making an $SU(2)$ dynamical study of the N^* as a resonance in the lowest mass (πN) channel alone quite appropriate. It is then possible to argue from either of two viewpoints that the electromagnetic mass splitting between adjacent charge states of the N^* is less than the 2.8 MeV predicted in Sec. II [Eq. (2.10)] on the basis of octet enhancement. In the first viewpoint, we recall the result of Sec. V that the

[39] Recently, the electromagnetic mass splittings within the $J=\frac{3}{2}^+$ decuplet have taken on some experimental interest with the measurements of $M(N^{*++})-M(N^{*-})=-0.6\pm 5.0$ MeV by G. Gidal, A. Kernan, and S. Kim, Lawrence Radiation Laboratory Report UCRL-11543 (to be published in *Proceedings of the 1964 International Conference on High Energy Physics, Dubna, U.S.S.R.*), $M(Y_1^{*-})-M(Y_1^{*+})=4.4\pm 2.2$ MeV by D. O. Huwe, Lawrence Radiation Laboratory Report UCRL-11291, 1964 (unpublished) and $M(Y_1^{*-})-M(Y_1^{*+})=17\pm 7$ MeV by W. A. Cooper, H. Filthuth, A. Friedman, E. Malamud, E. S. Gelsema, J. C. Kluyver, and A. G. Tenner, Phys. Letters **8**, 365 (1964).

TABLE XI. $SU(2)$ bootstrap: contributions to A_1 from shifts in exchanged masses (Ref. 41).

	N exch	N^* exch
N	$-\frac{1}{9}$	$-\frac{16}{9\sqrt{2}}\frac{\gamma_{33}}{\gamma_{11}}\left(\frac{D_{11}^2(M^x)}{M^x-M}\right)_c'$
N^*	$-\frac{4\sqrt{2}}{9}\frac{\gamma_{11}}{\gamma_{33}}\left(\frac{D_{33}^2(M)}{M-M^{33}}\right)'$	$-\frac{1}{9}\left(\frac{D_{33}^2(M^x)}{M^x-M^{33}}\right)_c'$

$SU(3)$ symmetric prediction for the N^* mass splittings comes mainly from the mass splitting of the "external baryons." Now the $\Sigma^+-\Sigma^0-\Sigma^-$ mass splittings are considerably larger than the n-p mass splitting. Therefore, most of the "external mass shift" contribution to δM^{N^*} is lost when strong symmetry breaking is taken into account and N^* is considered as a purely πN resonance. In the second viewpoint, we consider *starting* with an $SU(2)$ symmetric model of the N^* as a πN resonance. Now Abers, Zachariasen, and Zemach[12] have shown that whereas an $SU(3)$ symmetric bootstrap may be unstable, various model $SU(2)$ symmetric bootstraps are stable against small symmetry-violating perturbations. In terms of eigenvalues of the A matrix, this means that whereas A_8 may have an eigenvalue near one, the elements of A connecting $I=1$, $I=2$, and other $SU(2)$ violations have no eigenvalue near one. We show in the next section that this result, obtained by Abers, Zachariasen, and Zemach for $N-\pi$ and $\rho-\pi$ reciprocal bootstraps, also holds for the $N-N^*$ reciprocal bootstrap. The implication for N^* electromagnetic splittings is that when strong symmetry breaking is taken into account and the N^* treated as a separate $SU(2)$ multiplet, its electromagnetic driving terms are no longer strongly enhanced by the factor $(1-A)^{-1}$, and thus the electromagnetic splittings are not so large.

TABLE XII. $SU(2)$ bootstrap: contributions to A_1 from shifts in external masses.

	N ext	N^* ext
N	$1-A_1^{NN\text{ exch}}-\sqrt{2}A_1^{NN^*\text{ exch}}$	0
N^*	$\sqrt{2}-A_1^{N^*N\text{ exch}}-\sqrt{2}A_1^{N^*N^*\text{ exch}}$	0

Another case where higher order effects must certainly be taken into account is the pi-meson electromagnetic mass splitting, which is pure $I=2$ [an $I=1$ splitting would transform like the third component of isotopic spin and thus distinguish between $M(\pi^+)$ and $M(\pi^-)$, in violation of charge conjugation invariance]. Since $I=2$ occurs in the **27** representation but not in **8**, the observed $M(\pi^+)-M(\pi^0)$ splitting indicates that the **27** representation plays a greater role in the electromagnetic than in the strong mass splittings. It is likely that the explanation will be found in the strong symmetry breaking which splits the π off from the rest of the $J=0^-$ octet; when the π is treated as a separate $SU(2)$ multiplet, its mass splittings no longer need to lie so closely along the enhanced octet eigenvector.

TABLE XIII. Diagonal elements of mass matrices for the $J=\frac{3}{2}^+$ N^*.

Representation	N^{*++}	N^{*+}	N^{*0}	N^{*-}	Normalization factor
1	1	1	1	1	2
3	3	1	-1	-3	$\sqrt{20}$
5	1	-1	-1	1	2
7	1	-3	3	-1	$\sqrt{20}$

TABLE XIV. Diagonal elements of mass matrices for the nucleon N.

Representation	p	n	Normalization factor
1	1	1	$\sqrt{2}$
3	1	-1	$\sqrt{2}$

The neutron-proton electromagnetic mass difference is also appropriately treated by an $SU(2)$ calculation involving only the lowest mass (πN) channel. In this case, the result is not so different from the $SU(3)$ calculation because even in $SU(3)$, with the customary values for the F over D ratio, the nucleon has about 70% probability to occur in the πN channel. It happens that the enhanced octet eigenvector does not determine the neutron-proton mass difference very well, however, for another reason: the enhanced eigenvector involves the symmetric and antisymmetric octet baryon mass matrices in the combination $(\mathbf{8}_a-0.25\,\mathbf{8}_s)$, whereas the

TABLE XV. Ratios between $A_r^{N^*N\text{ ext}}$ and $A_1^{N^*N\text{ ext}}$.

Representation of δM^{N^*}	Representation of $\delta M^{N\text{ ext}}$ 1	3
1	1	0
3	0	$\frac{1}{3}\sqrt{5}$
5	0	0
7	0	0

neutron-proton mass difference involves the combination $\mathbf{8}_a+(3/\sqrt{5})\mathbf{8}_s$. The near orthogonality of the two combinations explains why the neutron-proton mass difference is relatively small, but also implies that the nonenhanced terms associated with other eigenvectors play a larger role than usual.

Although we have stressed cases where $SU(3)$ symmetry is rather badly broken in this section, it should be kept in mind that usually the symmetry is more nearly obeyed and first-order calculations should give the essential features correctly.

VII. CALCULATION OF THE A MATRIX FOR N AND N^* MASS SHIFTS

In the preceding section we found that owing to strong $SU(3)$ symmetry breaking, an $SU(2)$-symmetric reciprocal bootstrap for N and N^* in the πN channel is likely to provide more accurate electromagnetic mass shifts than an $SU(3)$-symmetric reciprocal bootstrap. In the $SU(2)$ calculation, the higher mass channels such as $K\Sigma$ are relegated to the status of small corrections. Thus motivated, we proceed in the present section to set up the A matrix for the $N-N^*$ reciprocal bootstrap. Actually the matrix element A^{NN} has already been given in Dashen's calculation of the neutron-proton mass difference,[20] but in the present treatment we shall give a fuller derivation and indicate the uncertainties in the analysis.

TABLE XVI. Ratios between $A_r^{NN\text{ ext}}$ and $A_1^{NN\text{ ext}}$.

Representation of δM^N	Representation of $\delta M^{N\text{ ext}}$ 1	3
1	1	0
3	0	$-\frac{1}{3}$

We begin with a review of the $SU(2)$-symmetric reciprocal bootstrap model for N and N^* as first presented by Chew.[40] One considers πN scattering, with N and N^* poles appearing in the direct channel and N and N^* exchanges in the crossed channel. As in the $SU(3)$ case, the N and N^* exchanges are represented by nearby pseudopoles at $W=M$ and $W=M^x$, respectively. The P-wave scattering amplitudes are defined by Eq. (5.1) as in the $SU(3)$ case. The $J=\frac{3}{2}^+$ amplitude has a direct channel pole $-\gamma_{33}/(W-M^{33})$ and N and N^* exchange poles $(4/9)\gamma_{11}/(W-M)$ and $\frac{1}{9}\gamma_{33}/(W-M^x)$, respectively. The $J=\frac{1}{2}^+$ amplitude has a direct channel pole $-\gamma_{11}(W-M)$ and exchange poles $\frac{1}{9}\gamma_{11}(W-M)$ and $(16/9)\gamma_{33}/(W-M^x)$. Here $\gamma_{11}=3f_{\pi NN}^2/(M^\pi)^2\approx 0.24/(M^\pi)^2$, and the reciprocal bootstrap provides the relation $\gamma_{33}\approx\frac{1}{2}\gamma_{11}$ in agreement with experiment.

TABLE XVII. Ratios between $A_r^{N^*N^*\text{ exch}}$ and $A_1^{N^*N^*\text{ exch}}$.

Representation of δM^{N^*}	Representation of $\delta M^{N^*\text{ exch}}$ 1	3	5	7
1	1	0	0	0
3	0	$-\frac{1}{3}$	0	0
5	0	0	-1	0
7	0	0	0	3

Again we shall label the elements of the A matrix by the dimensions of the irreducible representations they connect, viz. A_1 for $I=0$ mass shifts, A_3 for $I=1$ mass shifts, and so forth. Exchange contributions to A_1 are calculated from the $SU(2)$ symmetric bootstrap in the same manner as before; the results are presented in Table XI.[41] Invariance under changes in the mass scale then gives the external mass contributions to A_1, which are presented in Table XII.

[40] G. Chew, Phys. Rev. Letters **7**, 394 (1961).

TABLE XVIII. Ratios between $A_r^{NN^*\text{ exch}}$ and $A_1^{NN^*\text{ exch}}$.

Representation of δM^N	Representation of $\delta M^{N^*\text{ exch}}$ 1	3	5	7
1	1	0	0	0
3	0	$-\frac{1}{3}\sqrt{5}$	0	0

To compute the elements of A_3, A_5, and A_7, we need normalized mass matrices transforming like $I=1$, 2, and 3. It is most convenient, for each value of I, to use the matrix which is diagonal in electric charge. The appropriate matrices are given in Tables XIII and XIV. Note that the different matrices for the N^* are orthogonal, as they should be, and similarly for the N matrices. The **1** and **3** matrices will be recognized as just the unit matrix and I_3, multiplied by normalization factors, and the **5** matrix is simply proportional to $I_3^2-\mathrm{Tr}(I_3^2)/\mathrm{Tr}(1)$.

TABLE XIX. Ratios between $A_r^{N^*N\text{ exch}}$ and $A_1^{N^*N\text{ exch}}$.

Representation of δM^{N^*}	Representation of $\delta M^{N\text{ exch}}$ 1	3
1	1	0
3	0	$-\frac{1}{3}\sqrt{5}$
5	0	0
7	0	0

Finally, the mass matrices are used as before to compute the ratios of external and exchange parts of A_3, A_5, and A_7 to A_1. The ratios are given in Tables XV–XX.

From the information in Tables XV–XX, all elements of the A matrix connecting δM^N and δM^{N^*} can be computed. If we take $c=0.6$ and

$$D=-(4/3)M(W-M)/(W-(7/3)M) \qquad (7.1)$$

as recommended in Appendix D, we find the eigenvalues $A_1=1.00$ and -0.21, $A_3=-0.61$ and 0.35, $A_5=0.01$,

TABLE XX. Ratios between $A_r^{NN\text{ exch}}$ and $A_1^{NN\text{ exch}}$.

Representation of δM^N	Representation of $\delta M^{N\text{ exch}}$ 1	3
1	1	0
3	0	$-5/3$

[41] The N^* exchanges have been multiplied by a factor c to take corrections to the narrow resonance approximation into account. These corrections are estimated in Appendix D. They are greater for N^* than for the other members of the decuplet since N^* is broadest.

and $A_7=-0.04$. Apart from the unit eigenvalue of A_1 which represents the effect of mass scale invariance, no eigenvalue of A lies near one. This result continues to hold as c is varied up to 1 and as D is varied from the curved form (7.1) to the straight line dependence $D=(W-M)$. Thus the $SU(2)$, $N-N^*$ bootstrap is stable against small perturbations, as Abers, Zachariasen, and Zemach[12] have already shown for several other reciprocal bootstrap models. It is also implied that no single eigenvector dominates, so a full study of the driving terms must be made to get information about the pattern as well as the scale of electromagnetic mass splitting. A study of this kind has been carried out by Dashen for the neutron-proton mass difference, but not yet for the N^* splittings.

Dashen made use of the matrix elements $A_3{}^{NN\text{ exch}}$, $A_3{}^{NN\text{ ext}}$, and $A_3{}^{NN^*\text{ exch}}$ in his calculation of the neutron-proton mass difference,[20] without providing a full derivation for them. The present treatment shows explicitly how they are derived, and what uncertainties they are subject to. The value

$$A_3{}^{NN\text{ exch}}=5/27 \tag{7.2}$$

used by Dashen can easily be obtained from Tables XI and XX. It is independent of the curvature of the D function. The quantity $A_3{}^{NN\text{ ext}}$ can be obtained from Tables XII and XVI. Dashen's estimate $A_3{}^{NN\text{ ext}}\approx-\frac{1}{3}$ was based on the approximation that $A_1{}^{NN\text{ ext}}\approx-1$, which is valid if $A_1{}^{NN^*\text{ exch}}$ is small compared to one. Actually, the most reasonable values of D and c give $A_3{}^{NN\text{ ext}}$ between -0.4 and -0.5, which would lead to a 5 or 10% correction to Dashen's result for the neutron-proton mass difference.

Dashen also estimated that mass shifts in the exchanged N^* have only a small effect on the neutron-proton mass shift. This estimate involves the product of $A_3{}^{NN^*\text{ exch}}$, which can be obtained from Tables XI and XVIII, and δM^{N^*}. The latter can be bounded either by the theory of Sec. II, Eq. (2.10)

$$M(N^{*-})-M(N^{*++})=8.4\text{ MeV} \tag{7.3}$$

(see Sec. VI for arguments indicating that strong symmetry breaking will reduce this estimate) or by the experiment of Gidal *et al.*[39]:

$$M(N^{*-})-M(N^{*++})=0.6\pm5.0\text{ MeV}. \tag{7.4}$$

If we take the reasonable values $c=0.6$, $D=-(4/3)\times M(W-M)/[W-(7/3)M]$, and $M(N^{*-})-M(N^{*++})<5$ MeV, we find that the effect on the neutron-proton mass difference is certainly less than 20%, and this was the basis for Dashen's statement that the effect is small.

APPENDIX A

In this Appendix we derive the asymptotic behavior of the D function in potential theory. We define D to have no poles or zeros on the physical sheet, except for the zeros at bound states. With this definition, the D function for a channel containing n bound states can be written in the Omnes form[42]

$$D(q^2)=C\prod_{i=1}^{n}(q^2-q_{B_i}{}^2)\exp\left[-\frac{(q^2-q_0{}^2)}{\pi}\times\int_0^\infty\frac{dq'^2\eta(q'^2)}{(q'^2-q_0{}^2)(q'^2-q^2)}\right]. \tag{A1}$$

Here, q is the momentum, q_{B_i} the momentum at the ith bound state, q_0 the momentum at an arbitrary subtraction point where D can be normalized, and η is the phase shift. We define $\eta(0)=0$. At large q^2, D behaves like

$$\lim_{q^2\to\infty}D(q^2)=Cq^{2n}\exp\left[\frac{\eta(\infty)}{\pi}\ln q^2+\text{const}\right]\sim q^{2(n+\eta(\infty)/\pi)}. \tag{A2}$$

In potential theory, Levinson's theorem[43] tells us that

$$[\eta(\infty)-\eta(0)]/\pi=N-n. \tag{A3}$$

The right side of this equation represents the number (N) of stable particles before the interactions are turned on (we interpret these as elementary particles) minus the number (n) of stable particles after the interactions are turned on. Combining Eqs. (A2) and (A3), and recalling that $\eta(0)$ vanishes by definition, we obtain the asymptotic behavior

$$D(q^2)\sim q^{2N}\quad\text{as}\quad q^2\to\infty. \tag{A4}$$

APPENDIX B

In this appendix, we present arguments which indicate that δM^{Π}, $\delta\Gamma^{B\Delta\Pi}$, and $\delta\Gamma^{BB\Pi}$ may have a relatively small effect on δM^B and δM^Δ.

First, we consider the effect of shifts in M^{Π}. These shifts can affect the dispersion relations for δM^B and δM^Δ by altering the right- and left-hand cuts of the ΠB scattering amplitude. The Π appears only as an "external particle" in our model, so the alterations are purely kinematic. On the left cut, if M^{Π} is set equal to zero, the nearby B and Δ exchange singularities shrink to pseudopoles at $W=M^B$ and $W=2M^B-M^\Delta$ respectively. The corrections, both in the location and magnitude of the left-hand singularities, due to nonzero M^{Π}, are all of order $(M^{\Pi}/M^B)^2$. For example, the pseudopole at $W=M^B$ due to B exchange is really[30,31] a "short cut" running from $W=M^B[1-(M^{\Pi}/M^B)^2]^{1/2}$ to $W=M^B[1+2(M^{\Pi}/M^B)^2]^{1/2}$. Therefore, shifts in M^{Π} on the left cut have an effect of order $\delta M^{\Pi}(M^{\Pi}/M^B)$ on δM^B and δM^Δ. On the right cut, our dispersion integrals for δM^B and δM^Δ depend only on $\delta\rho$,[11] where ρ is the

[42] R. Omnes, Nuovo Cimento 8, 316 (1958).
[43] N. Levinson, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **25**, No. 9 (1949).

kinematic factor in the ΠB amplitude $T=(e^{2i\eta}-1)/\rho$. We take[20]

$$\rho=\frac{(M^B)^2q[(W-M^B)^2-(M^\Pi)^2]}{W^2}, \tag{B1}$$

where

$$q^2=\frac{[(W+M^B)^2-(M^\Pi)^2][(W-M^B)^2-(M^\Pi)^2]}{4W^2}, \tag{B2}$$

and the result of plugging $\delta\rho$ into the dispersion integral for δM^B or δM^Δ is again a contribution of order $\delta M^\Pi(M^\Pi/M^B)$.

Comparing our result

$$\delta M^B\approx(M^\Pi/M^B)\delta M^\Pi+\cdots \tag{B3}$$

with

$$\delta M^B/M^B=A^{B\Pi}(\delta M^\Pi/M^\Pi)+\cdots, \tag{B4}$$

we see that $A^{B\Pi}$ (or $A^{\Delta\Pi}$) is or order $(M^\Pi/M^B)^2$ for our particular definition of the A coefficients. The contribution of $A^{B\Pi}$ (or $A^{\Delta\Pi}$) in the conditions expressing the invariance of bootstrap theory under a change of over-all mass scale,[7] e.g.,

$$1=A_1^{B\Pi}+A_1^{BB}+\cdots, \tag{B5}$$

is also of order $(M^\Pi/M^B)^2$. We have not made a thorough investigation of the coefficients of these various factors (M^Π/M^B). Nevertheless, the circumstance that all contributions of δM^Π to δM^B and δM^Δ are down by powers of (M^Π/M^B) makes it plausible that $A^{B\Pi}$ and $A^{\Delta\Pi}$ are relatively small, and we assume this in the discussion of Sec. IV.

Next we consider the effect on δM^B and δM^Δ of shifts in the $B\Delta\Pi$ and $BB\Pi$ couplings. For our present purposes, it is sufficient to consider the one-channel formula

$$\delta M^B=\frac{1}{2\pi iR^B}\int\frac{D^2(W')\delta T(W')}{(W'-M^B)}dW', \tag{B6}$$

where C is a contour traversed clockwise around the left-hand singularities. The left-hand singularities of T in our model are the B and Δ exchange poles:

$$T\approx R^B(M^B-W)+R^\Delta/(M^x-W). \tag{B7}$$

Here $M^x=2M^B-M^\Delta$. Now variations on the coupling have the effect

$$\delta T\approx\delta R^B/(M^B-W)+\delta R^\Delta/(M^x-W). \tag{B8}$$

Plugging the effect of these coupling shifts into Eq. (B6), we note that the double zero of $D^2(W)$ cancels the double pole at $W=M^B$, so that δM^B receives no contribution from the shift in $BB\Pi$ coupling. The $B\Delta\Pi$ coupling shift contributes

$$\delta M^B\approx[D^2(M^x)/(M^x-M^B)](\delta R^\Delta/R^B). \tag{B9}$$

For purposes of making a rough estimate, let $D(W)\approx(W-M^B)$ in the low-energy region (this gives an overestimate since D is expected to increase somewhat

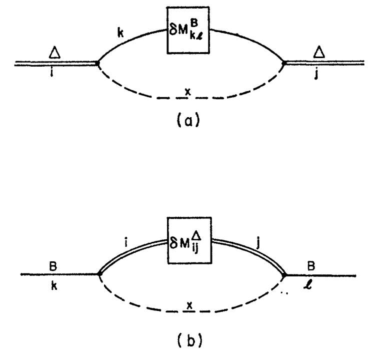

FIG. 1. Diagrams representing the group-theoretical structure of (a) the effect a shift in external baryon mass δM^B has on δM^Δ, (b) the effect a shift in external Δ mass δM^Δ would have on δM^B, if the $\Pi\Delta$ channel were included in a model of B. The pseudoscalar meson octet is represented by a dotted line, the baryon octet by a solid line, and Δ by a double solid line.

more slowly than linearly). Then we have

$$\frac{\delta M^B}{M^B}\approx\left[\frac{M^x-M^B}{M^B}\right]\frac{\delta R^\Delta}{R^B}=\left[\frac{M^B-M^\Delta}{M^B}\right]\frac{\delta R^\Delta}{R^B}. \tag{B10}$$

The R's are related to the dimensionless Γ's by a dimensional factor, and if, for example, we take the same dimensional factor for both R^B and R^Δ, we obtain

$$\frac{\delta M^B}{M^B}\approx\left[\frac{M^B-M^\Delta}{M^B}\right]\frac{\delta\Gamma^{B\Delta\Pi}}{\Gamma^{BB\Pi}}. \tag{B11}$$

A similar study of the reciprocal effect, of δM^B on δR^Δ, can be made which gives no indication that $\delta\Gamma^{B\Delta\Pi}/\Gamma^{BB\Pi}$ should exceed $\delta M^B/M^B$. Thus it appears likely that the effect of $\delta\Gamma^{B\Delta\Pi}$ on δM^B is of order $(M^B-M^\Delta)/M^B$. The same factor appears in the effect of $\delta\Gamma^{BB\Pi}$ on δM^Δ. The effect of $\delta\Gamma^{B\Delta\Pi}$ on δM^Δ is smaller, due to the well-known fact that Δ exchange has a small coefficient in the Δ channel.

The factor $(M^\Delta-M^B)/M^B$ which appears in the effect of vertex shifts on δM^B and δM^Δ makes it plausible, though not certain, that these effects are relatively small, and we assume this in the discussion of Sec. IV.[44]

APPENDIX C

In this appendix, we discuss some symmetry properties of the A matrix.

For concreteness, we begin by considering the contribution to $A^{\Delta B}$ from external B mass shifts. We recall from Ref. 11 that this contribution [Fig. 1(a)] has the

[44] A contrary opinion on this important question is stated by R. Cutkosky and P. Trajanne, Phys. Rev. **133**, B1292 (1964). The difference can be traced back to the convergence of the dispersion integrals. The numerator of the factor $(M^\Delta-M^B)/M^B$ is just the distance from the direct channel pole to the left-hand singularity. This distance is small in our case because of our assumption of good convergence. If the formalism converges less rapidly, the distance can grow; therefore, the factor grows, and one reaches the contrary conclusion that vertex shifts may have a large effect on δM^B and δM^Δ [R. Cutkosky (private communication)].

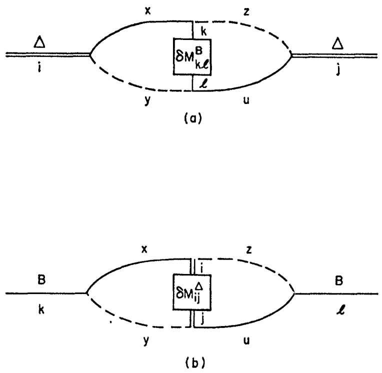

FIG. 2. Diagram representing the group-theoretical structure of (a) the effect a shift in exchanged B mass has on δM^{Δ}, and (b) the effect a shift in exchanged Δ mass has on δM^{B}.

structure

$$A_{ij,kl}{}^{\Delta B}=K_{\text{ext}}{}^{\Delta B}C_{ij,kl}{}^{\Delta B}, \tag{C1}$$

$$C_{ij,kl}{}^{\Delta B}=\sum_x g^{ikx}g^{jlx}, \tag{C2}$$

where K is a single number, g^{ikx} is the Clebsch-Gordan coefficient at the $\Delta^i B^k \Pi^x$ vertex, i and $j=1, \cdots 10$, and k and $l=1, \cdots 8$.

Similarly, if we suppose that B couples partly to the $\Delta\Pi$ channel, the "external mass shift" contribution to $A^{B\Delta}$ [Fig. 1(b)] has the structure

$$A_{kl,ij}{}^{B\Delta}=K_{\text{ext}}{}^{B\Delta}C_{kl,ij}{}^{B\Delta}, \tag{C3}$$

$$C_{kl,ij}{}^{B\Delta}=\sum_x g^{ikx}g^{jlx}. \tag{C4}$$

In Eq. (C4), remember that the first index of g^{ikx} refers to Δ and the second to B.

Now it is clear that the A matrix is not symmetric; for example, in the model employed in the present paper, both Δ and B occur in the $B\Pi$ channel but not in the $\Delta\Pi$ channel, so external B mass shifts influence Δ mass shifts ($K_{\text{ext}}{}^{\Delta B}\neq 0$) but there are no external Δ mass shifts to influence B mass shifts ($K_{\text{ext}}{}^{B\Delta}=0$). By comparison of Eq. (C2) with Eq. (C4), however, we see that the asymmetry is confined to K; the group theoretical factors $C^{\Delta B}$ and $C^{B\Delta}$ are just the transpose of one another:

$$C_{ij,kl}{}^{\Delta B}=C_{kl,ij}{}^{B\Delta}. \tag{C5}$$

Thus if we had included the $\Delta\Pi$ channel in our model, all components of $A^{\Delta B}$ would follow immediately from $A^{B\Delta}$ and a single component of $A^{\Delta B}$. It is evident that these results are not confined to Δ and B, but are completely general.[45]

Next we consider the contribution of *exchanged* mass shifts to $A^{\Delta B}$ in the same spirit. From Ref. 11, this contribution [Fig. 2(a)] has the structure

$$A_{ij,kl}{}^{\Delta B}=K_{\text{exch}}{}^{\Delta B}C_{ij,kl}{}'^{\Delta B}, \tag{C6}$$

$$C_{ij,kl}{}'^{\Delta B}=\sum_{xyzu} g^{ixy}G^{ylu}g^{juz}G^{zkx}, \tag{C7}$$

where g has the same significance as before and G is the Clebsch-Gordan coefficient at the $\Pi^y B^l B^u$ vertex.

Similarly, the "exchanged mass" contribution to $A^{B\Delta}$ [Fig. 2(b)] has the form

$$A_{kl,ij}{}^{B\Delta}=K_{\text{exch}}{}^{B\Delta}C_{kl,ij}{}'^{B\Delta}, \tag{C8}$$

$$C_{kl,ij}{}'^{B\Delta}=\sum_{xyzu} G^{ykx}g^{ixz}G^{zlu}g^{juy}. \tag{C9}$$

Again we have been careful to use the first index of g^{kxz} for Δ, the second for B, and so forth. Just as for the external mass term, $K_{\text{exch}}{}^{B\Delta}$ and $K_{\text{exch}}{}^{\Delta B}$ are different numbers—B exchange differs from Δ exchange both in range and coupling strength—so the A matrix is not symmetric. An interchange of the dummy indices y and z in (C9) suffices to show, however, that the group theoretical factor $C'^{B\Delta}$ is again just the transpose of $C'^{\Delta B}$:

$$C_{kl,ij}{}'^{B\Delta}=C_{ij,kl}{}'^{\Delta B}. \tag{C10}$$

Once again it is evident that these results are not confined to Δ and B, but are completely general.

To illustrate the practical use of Eq. (C10), let us replace (ij) and (kl) by irreducible representations of $SU(3)$, all normalized in the same way, as the set of states which C connects. Equation (C10) can then be expressed by

$$C_1{}'^{B\Delta}=C_1{}'^{\Delta B}, \tag{C11}$$

$$C_8{}'^{B_s\Delta}=C_8{}'^{\Delta B_s}, \tag{C12}$$

$$C_8{}'^{B_a\Delta}=C_8{}'^{\Delta B_a}, \tag{C13}$$

$$C_{27}{}'^{B\Delta}=C_{27}{}'^{\Delta B}, \tag{C14}$$

where the subscripts s and a refer to the symmetric and antisymmetric octets occurring in $8\otimes 8$. Equations (C11)–(C14) could be used in Sec. V to obtain $A_8{}^{B_s\Delta}$, $A_8{}^{B_a\Delta}$, and $A_{27}{}^{B\Delta}$ from $A^{\Delta B}$ and $A_1{}^{B\Delta}$.

As a special case of the relation

$$C_{ij,kl}{}^{\alpha\alpha'}=C_{kl,ij}{}^{\alpha'\alpha} \tag{C15}$$

that we have just proved, $C^{\alpha\alpha}$ is symmetric and therefore $A^{\alpha\alpha}$ is symmetric. For example, in Sec. V, the A coefficient connecting a symmetric octet mass shift $\delta M_{8(s)}{}^B$ to an antisymmetric octet mass shift $\delta M_{8(a)}{}^B$ is the same as the A coefficient connecting $\delta M_{8(a)}{}^B$ to $\delta M_{8(s)}{}^B$.

Finally one can prove that the A coefficients connecting mass shifts to coupling shifts, and coupling shifts to coupling shifts, also have a structure

$$A_{ij\cdots,kl\cdots}{}^{\alpha\alpha'}=K^{\alpha\alpha'}C_{ij\cdots,kl\cdots}{}^{\alpha\alpha'} \tag{C16}$$

such that the result

$$C_{ij\cdots,kl\cdots}{}^{\alpha\alpha'}=C_{kl\cdots,ij\cdots}{}^{\alpha'\alpha}, \tag{C17}$$

demonstrated above for the particular case of mass shifts, holds in general. The proof is similar to that for mass shifts, and will not be given here.

[45] Furthermore, Eq. (C5) applies even if A contains $SU(3)$ violations, as in the higher order effects of Sec. VI.

APPENDIX D

Each element of the A matrix depends on some dynamical parameters of the strongly interacting particles such as N and N^*. In some cases the parameters are well known, e.g., the nucleon mass and πNN coupling; in other cases they are less well known; e.g., the energy dependence of the D function for the nucleon channel. The present appendix contains a discussion of the element $A^{NN^*\,\mathrm{exch}}$, which is particularly sensitive to less well-known parameters. The discussion will give an idea of the uncertainties in our analysis. Although the uncertainties in the particular element $A^{NN^*\,\mathrm{exch}}$ are rather great, it should be kept in mind that "exchange" terms of A tend to be smaller than "external" terms, and the calculation of the neutron-proton mass difference, for example, is not actually very sensitive to $A^{NN^*\,\mathrm{exch}}$ (numerical estimates are given in Sec. VII).

The first uncertainty in $A^{NN^*\,\mathrm{exch}}$ arises from the factor

$$\frac{d}{dW}\left[\frac{D_{11}^2(W)}{(W-M^N)}\right]\Bigg|_{W=M^x} \tag{D1}$$

in Table XI. Here, D_{11} is the D function for the $J=\frac{1}{2}^+$, $I=\frac{1}{2}$ channel, and $M^x\approx 2M^N-M^{N^*}$ is the position of the N^* exchange pseudopole. We let D_{11} take the form

$$D_{11}(W)=(W-M^N)(M^N-M')/(W-M'), \tag{5.53}$$

where M' is a parameter which must be larger than M^N since D has singularities only on the right. An estimate for M' can be obtained by comparing (5.53) with the denominator function derived by Balázs.[46] Setting $M'=(7/3)M^N$ in (5.53) yields an expression which approximates Balázs' result to within a few percent throughout the range of interest. For this estimate of M', the factor (D1) takes on the value 0.38, whereas the factor would be one for a straight-line D function.

Another uncertainty, which is more unique to $A^{NN^*\,\mathrm{exch}}$, arises through the considerable departure of the $J=\frac{3}{2}^+$, $I=\frac{3}{2}$ resonance from a Breit-Wigner shape. In particular, the 3-3 resonance has a longer tail on the low-energy than on the high-energy side. Thus the simple pole approximation we have employed for N^* exchange must be modified. To do so, we return to the full expression for the contribution of the left cut to δM^N:

$$\delta M^N=\frac{-1}{\pi\gamma_{11}[D_{11}'(M^N)]^2}\int_L \frac{D_{11}^2(W')\,\mathrm{Im}\delta T_{11}(W')dW'}{W'-M^N} + \text{contribution from right cut.} \tag{D2}$$

The crossing matrix is such that $\mathrm{Im}\delta T_{11}$ receives a contribution (16/9) $\mathrm{Im}\delta T_{33}^x$ from exchange of N^*. The shift of the 3-3 amplitude in the crossed channel can in turn be expressed as

$$\delta T_{33}^x=\delta\left[\frac{e^{2i\eta_{33}}-1}{2i\rho}\right] =\frac{\delta\eta_{33}e^{2i\eta_{33}}}{\rho}-2\frac{\delta\rho}{\rho}T_{33}. \tag{D3}$$

If we drop the change in phase factor $\delta\rho$ (which involves shifts in external masses rather than the N^* exchange mass shift we wish to consider) and make the reasonable approximation that η_{33} is real in the resonance region, the imaginary part of δT_{33}^x becomes

$$\mathrm{Im}\delta T_{33}^x=(\delta\eta_{33}/\rho)\sin 2\eta_{33} \tag{D4}$$

so we have

$$\delta M^N=\frac{-16}{9\pi\gamma_{11}[D'(M^N)]^2}\times\int_L \frac{D_{11}^2(W')\delta\eta_{33}\sin(2\eta_{33})dW'}{\rho(W'-M^N)}. \tag{D5}$$

Evidently, $\sin(2\eta_{33})$ passes through zero at resonance, so the numerical value of the integral is determined by a competition between the regions above and below resonance. The factors are such that the region above dominates. Thus if we cut off the tail of the resonance on the high-energy side by letting η_{33} approach 180° more rapidly than in the Breit-Wigner formula, the integral decreases in value. This is the origin of the factor c in Sec. VII. A numerical integration of Eq. (D5), using η_{33} deduced from the shape of the physical resonance, yields the estimate $c\approx 0.6$.

[46] L. A. P. Balázs, Phys. Rev. **128**, 1935 (1962).